# CMSC 132: Object-Oriented Programming II

## Shortest Paths

# Quiz 1

One advantage of adjacency list representation over adjacency matrix representation of a graph is that in adjacency list representation, space is saved for sparse graphs.

A. True
B. False

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B. False

# Quiz 2

Traversal of a graph is different from tree because

A. There can be a loop in graph so we must maintain a visited flag for every vertex
B. DFS of a graph uses stack, but inorder traversal of a tree is recursive
C. BFS of a graph uses queue, but a time efficient BFS of a tree is recursive.
D. All of the above

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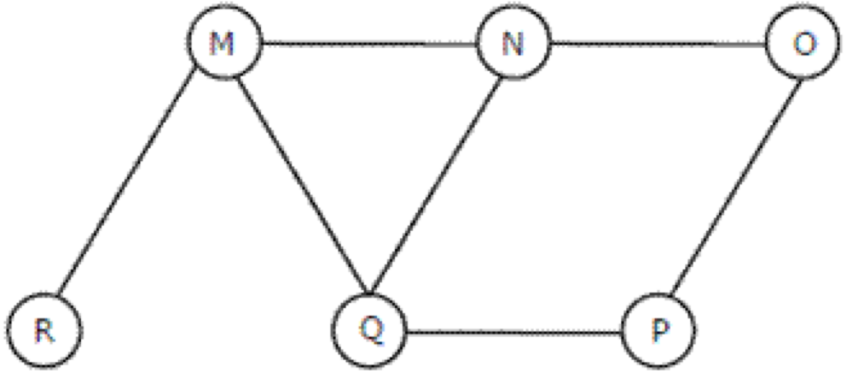
D. All of the above

# Quiz 3

One possible order of Breadth First Search on the following graph




A. MNOPQR
B. NQMPOR
C. QMNPRO
D. QMNPOR

# Quiz 3

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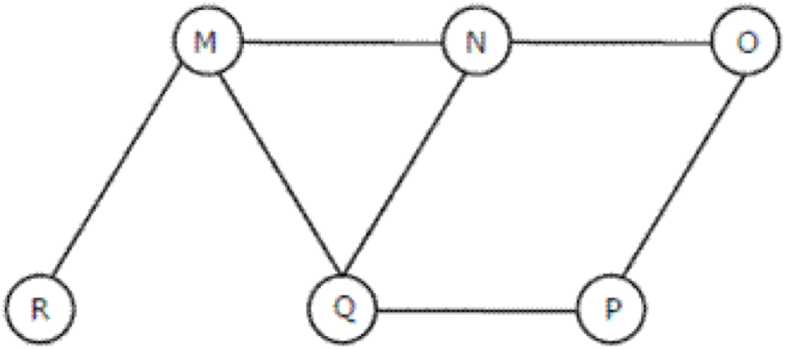


A. MNOPQR
B. NQMPOR
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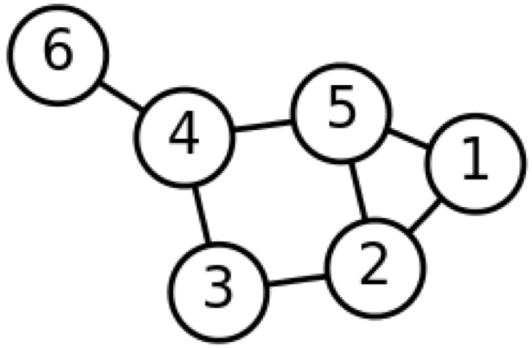
# Quiz 4

Given two vertices in a graph 1 and 6, which of the two traversals (BFS and DFS) can be used to find if there is path from 1 to 6?




A. Only BFS
B. Only DFS
C. Both BFS and DFS
D. Neither BFS nor DFS

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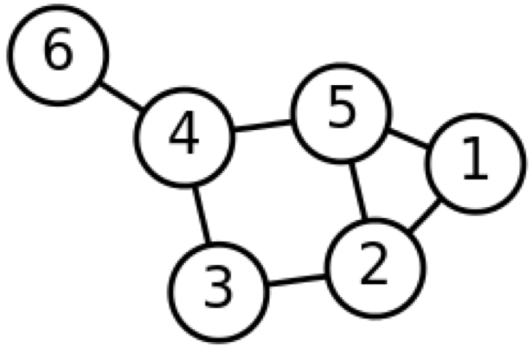
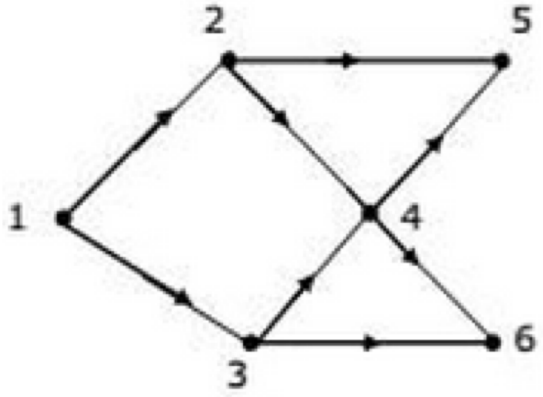



A. Only BFS
B. Only DFS
**C. Both BFS and DFS**
D. Neither BFS nor DFS

# Quiz 5

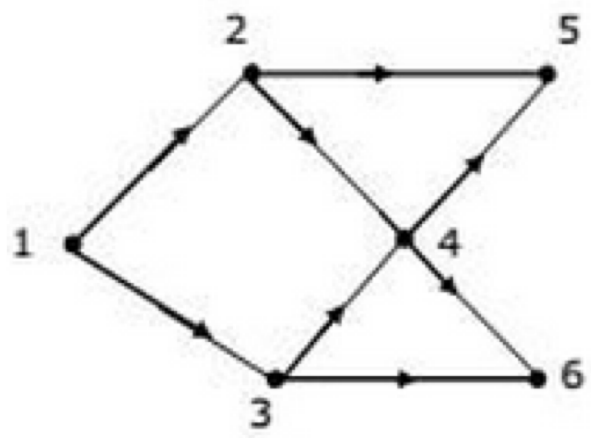
Consider the DAG with Consider V = {1, 2, 3, 4, 5, 6}, shown below. Which of the following is  NOT a topological ordering?




A. 1 2 3 4 5 6
B. 1 3 2 4 5 6
C. 1 3 2 4 6 5
D. 3 2 4 1 6 5

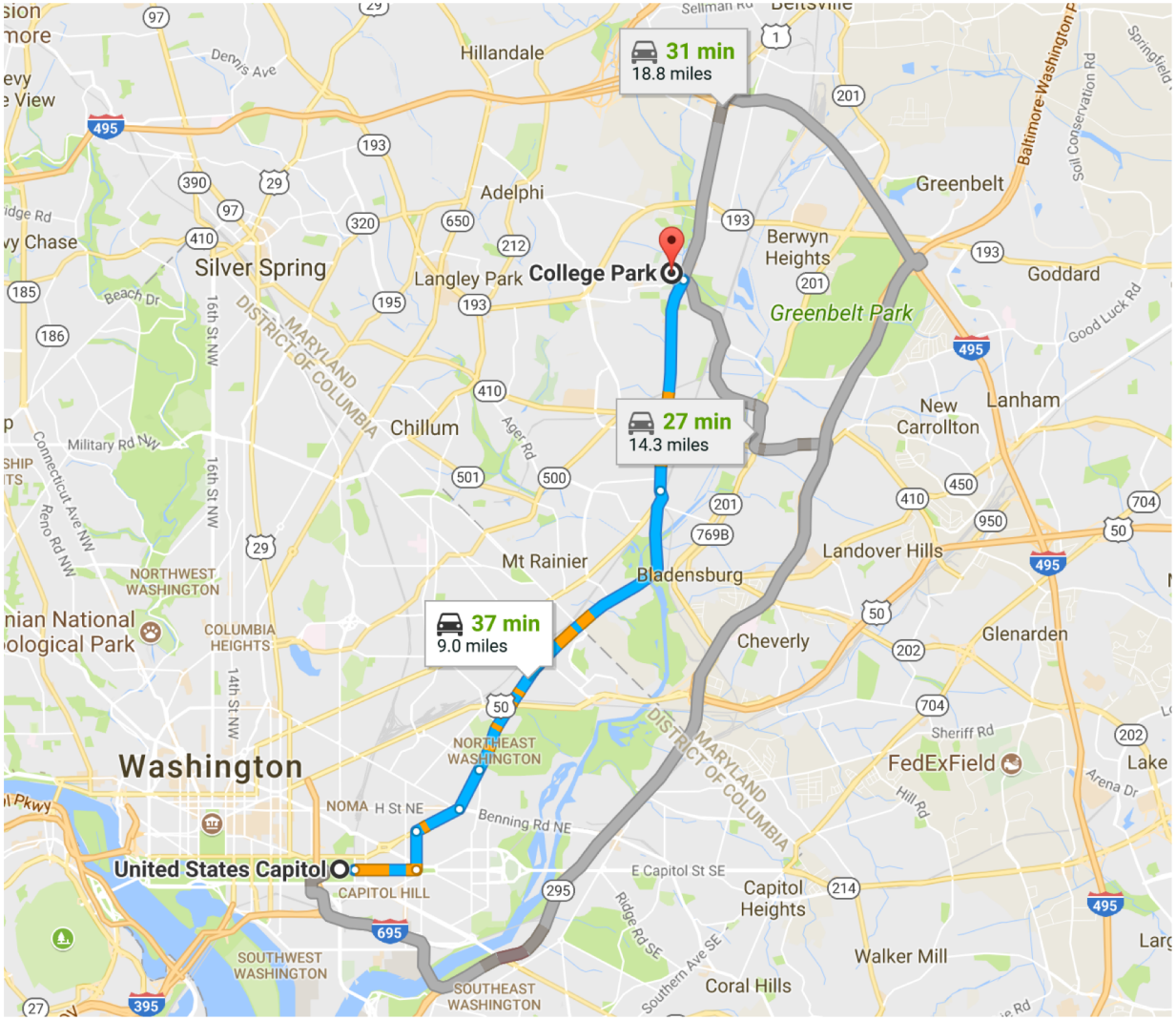
# Quiz 5

Consider the DAG with Consider V = {1, 2, 3, 4, 5, 6}, shown below. Which of the following is NOT a topological ordering?




A. 1 2 3 4 5 6
B. 1 3 2 4 5 6
C. 1 3 2 4 6 5
D. 3 2 4 1 6 5

# Shortest Paths

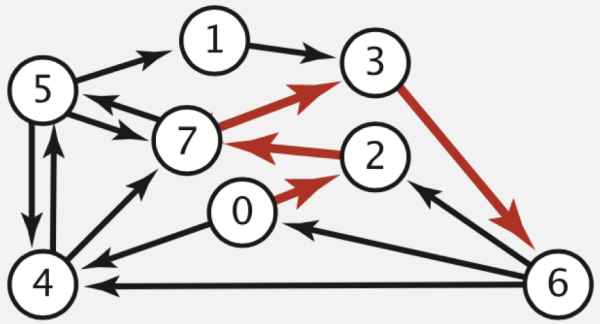


# Shortest paths

Given an edge-weighted digraph, find the shortest path from *s* to *t*.

**edge-weighted digraph**

```
4->5  0.35
5->4  0.35
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```




**shortest path from 0 to 6**

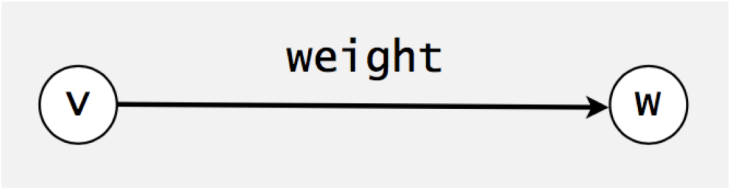
```
0->2  0.26
2->7  0.34
7->3  0.39
3->6  0.52
```

# Shortest path variants

- Which vertices?
  - Single source: from one vertex *s* to every other vertex.
  - Source-sink: from one vertex *s* to another *t*.
  - All pairs: between all pairs of vertices.
- Restrictions on edge weights?
  - Nonnegative weights.
  - Arbitrary weights.
- Cycles?
  - No directed cycles.
  - No "negative cycles."
- Simplifying assumption: Shortest paths from *s* to each vertex *v* exist.

# Weighted directed edge

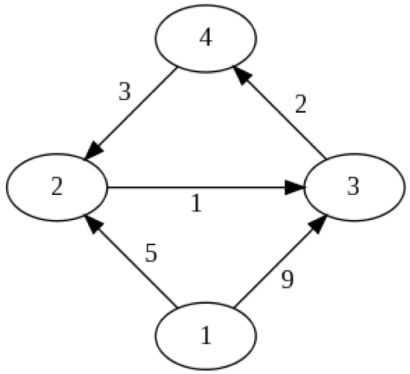
| | | |
|---|---|---|
| public class DirectedEdge | | |
| | DirectedEdge(int v, int w, double weight) | *weighted edge v→w* |
| int | from() | *vertex v* |
| int | to() | *vertex w* |
| double | weight() | *weight of this edge* |
| String | toString() | *string representation* |




Idiom for processing an edge e: **int v = e.from(), w = e.to();**

# Weighted directed edge implementation

```
public class DirectedEdge{
   private final int v, w;
   private final double weight;

   public DirectedEdge(int v, int w, double weight){
       this.v = v;
       this.w = w;
       this.weight = weight;
   }

   public int from() { return v; }
   public int to() { return w; }
   public double weight() { return weight; }
}
```



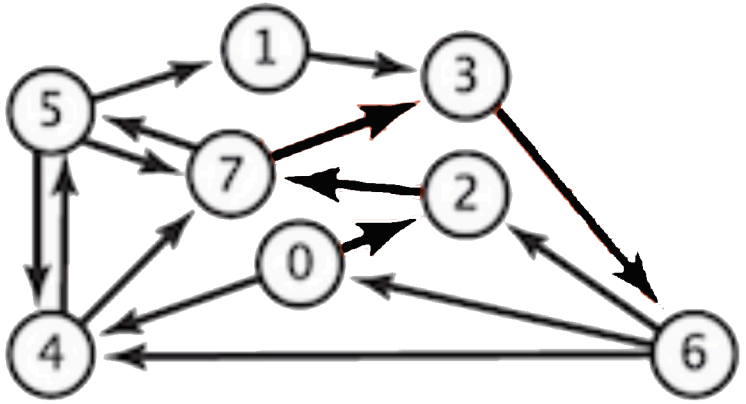
# Edge-weighted digraph

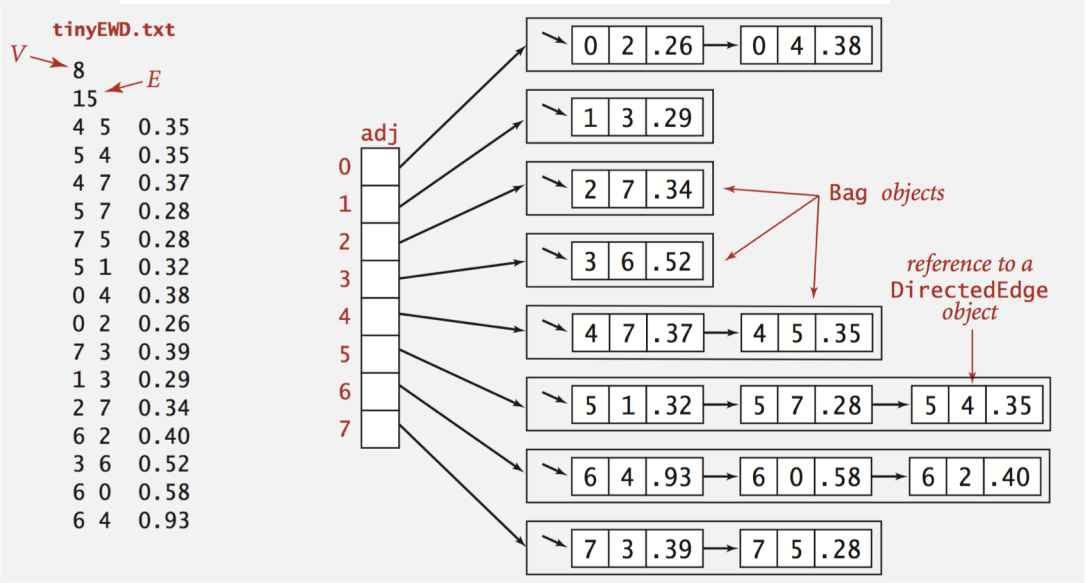
```
public class EdgeWeightedDigraph
```

| | | |
|---|---|---|
| | `EdgeWeightedDigraph(int V)` | *edge-weighted digraph with V vertices* |
| `void` | `addEdge(DirectedEdge e)` | *add weighted directed edge e* |
| `Iterable<DirectedEdge>` | `adj(int v)` | *edges pointing from v* |
| `int` | `V()` | *number of vertices* |
| `int` | `E()` | *number of edges* |
| `Iterable<DirectedEdge>` | `edges()` | *all edges* |
| `String` | `toString()` | *string representation* |

Conventions. Allow self-loops and parallel edges.

# Edge-weighted digraph: adjacency-lists representation



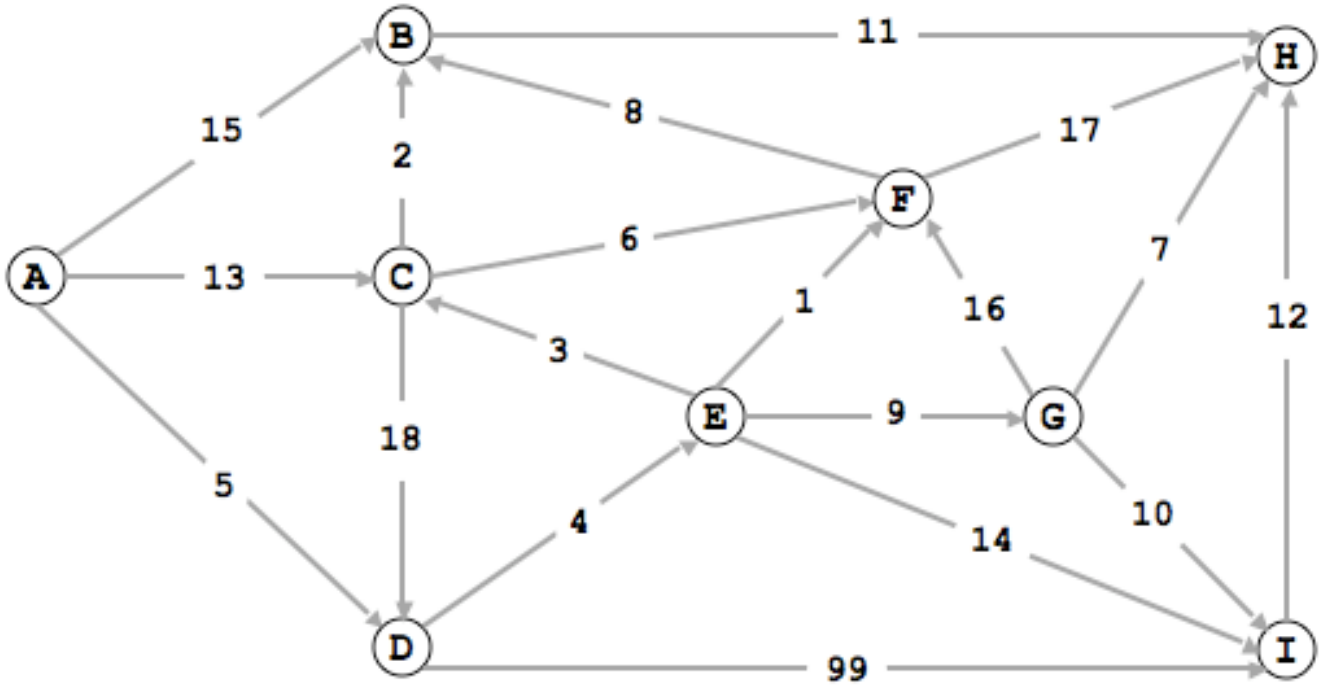



# Edge-weighted digraph implementation

```
public class EdgeWeightedDigraph{
  private final int V;
  private final Bag<DirectedEdge>[] adj;

  public EdgeWeightedDigraph(int V){
     this.V = V;
     adj = (Bag<DirectedEdge>[]) new Bag[V];
     for (int v = 0; v < V; v++)
         adj[v] = new Bag<DirectedEdge>();
  }

  public void addEdge(DirectedEdge e){
      int v = e.from();
      adj[v].add(e);
  }

  public Iterable<DirectedEdge> adj(int v){
       return adj[v];
  }
}
```

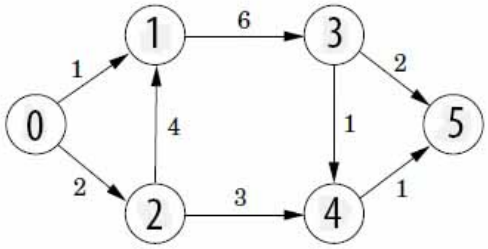
# Single-source shortest paths

What is the shortest distance and path from A to H?



# Single-source shortest paths

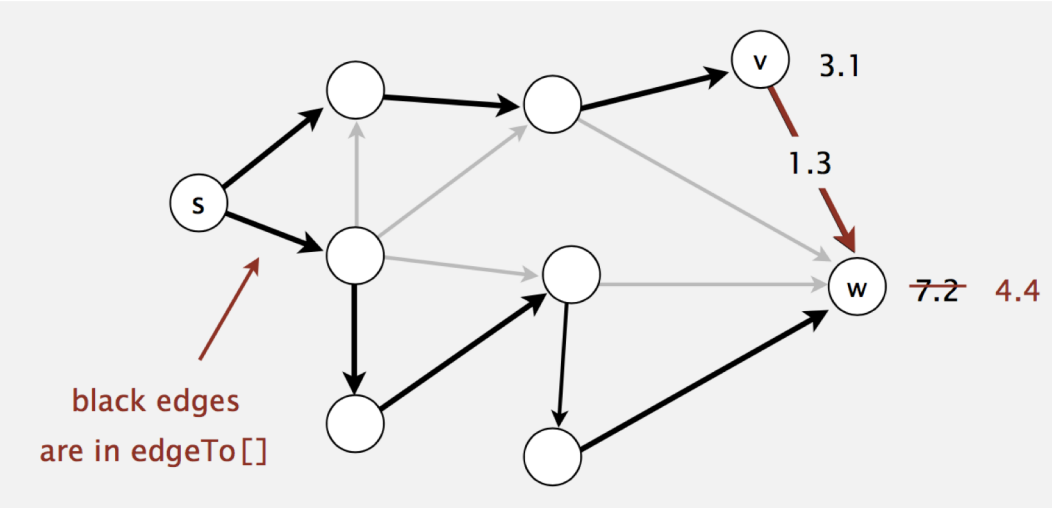
- Data structures: Represent the Shortest Path with two vertex-indexed arrays:
  - distTo[v] is length of shortest path from *s* to *v*.
  - edgeTo[v] is last edge on shortest path from *s* to *v*.




```
public double distTo(int v){
  return distTo[v];
}

public Iterable<DirectedEdge> pathTo(int v){
    Stack<DirectedEdge> path = new Stack<DirectedEdge>();
    DirectedEdge e = edgeTo[v];

    while (e != null){
      path.push(e);
      e = edgeTo[e.from()];
   }
   return path;
}
```

# Edge relaxation

- Relax edge e = v→w.
  - distTo[v] is length of shortest known path from **s to v**.
  - distTo[w] is length of shortest known path from **s to w**.
  - edgeTo[w] is last edge on shortest known path from s to w.
  - If **e = v→w** gives shorter path to w through v, update both distTo[w] and edgeTo[w]

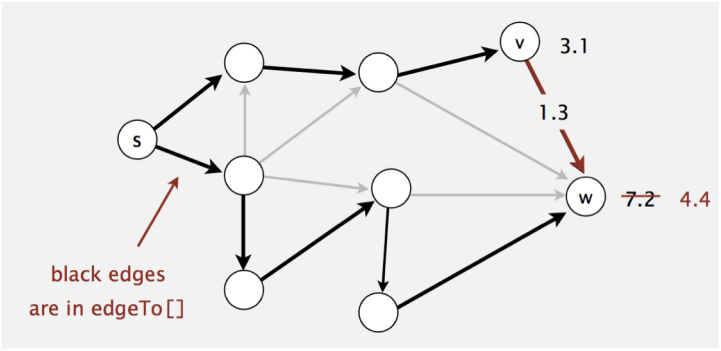



v→w successfully relaxes

# Edge relaxation

- Relax edge e = v→w.
  - distTo[v] is length of shortest known path from s to v.
  - distTo[w] is length of shortest known path from s to w.
  - edgeTo[w] is last edge on shortest known path from s to w.
  - If e = v→w gives shorter path to w through v, update both distTo[w] and edgeTo[w]

```
private void relax(DirectedEdge e) {
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight()) {
      distTo[w] = distTo[v] + e.weight();
      edgeTo[w] = e;
  }
}
```



# Generic shortest-paths algorithm

**Generic algorithm (to compute SPT from s)**

**Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.**

**Repeat until optimality conditions are satisfied:**

**Relax any edge.**

Efficient implementations: How to choose which edge to relax?

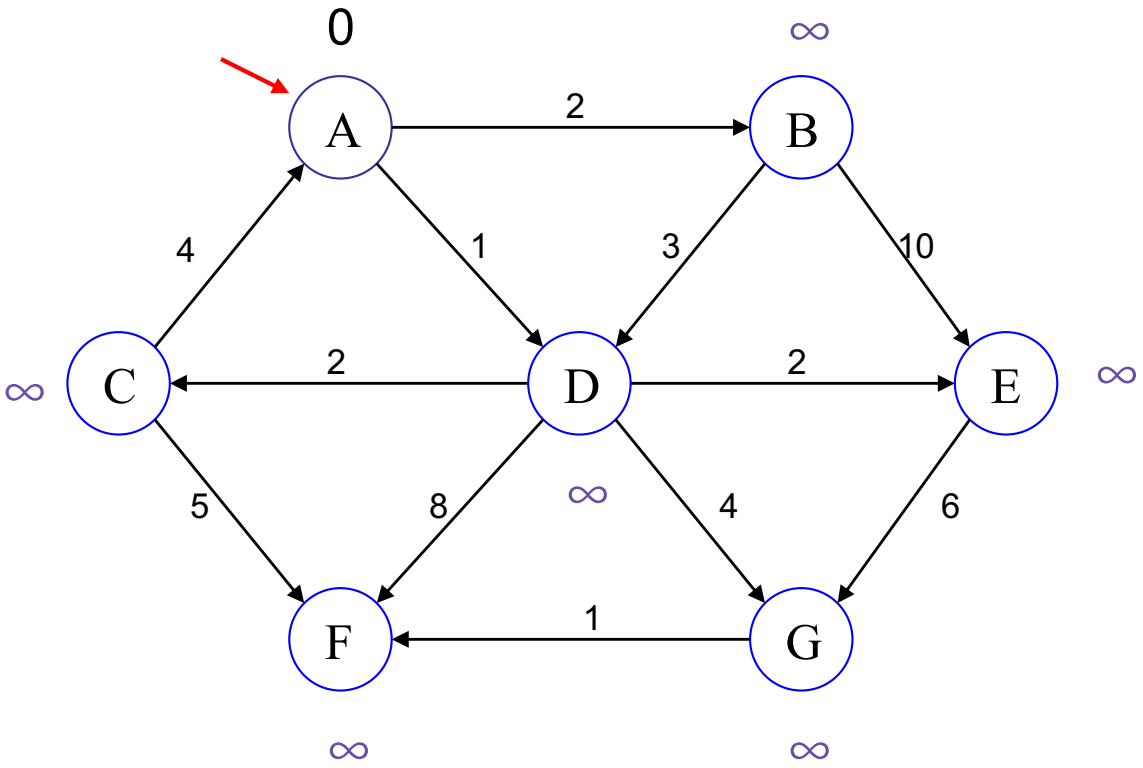
- Dijkstra's algorithm (nonnegative weights).
- Topological sort algorithm (no directed cycles).
- Bellman-Ford algorithm (no negative cycles).

# Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges pointing from that vertex.

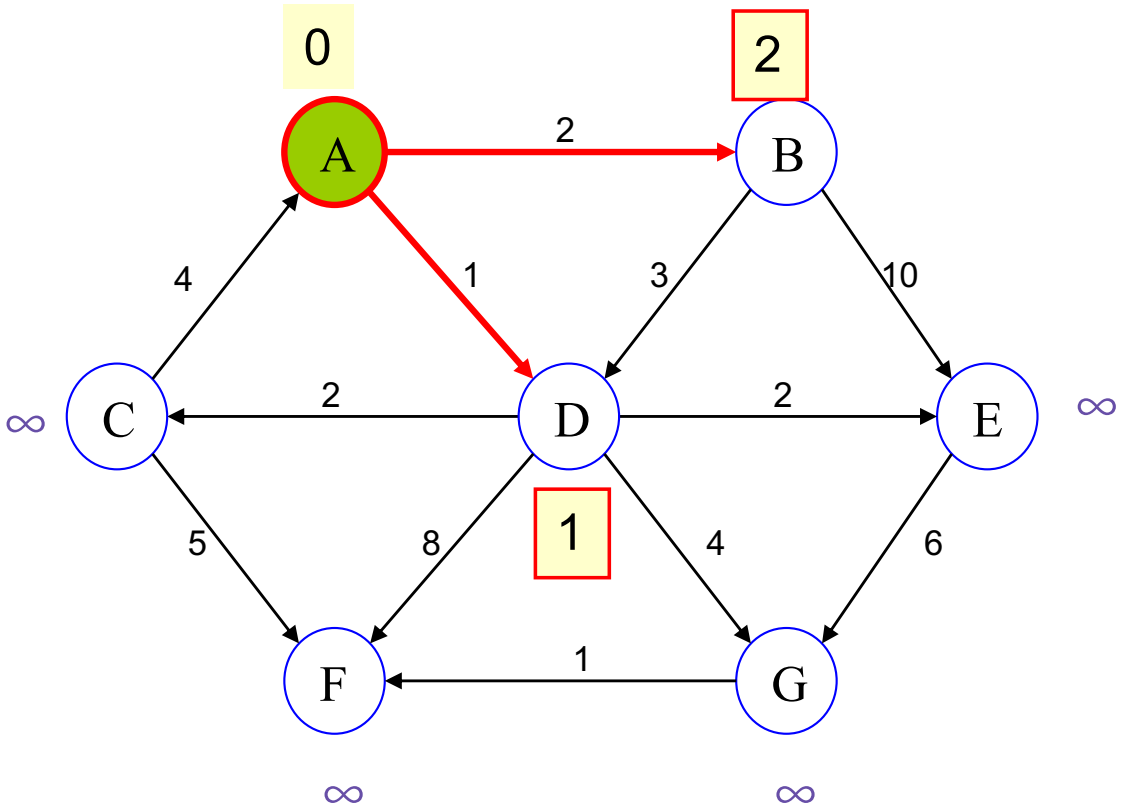
# Dijkstra's algorithm Demo

Pick vertex in List with minimum distance.



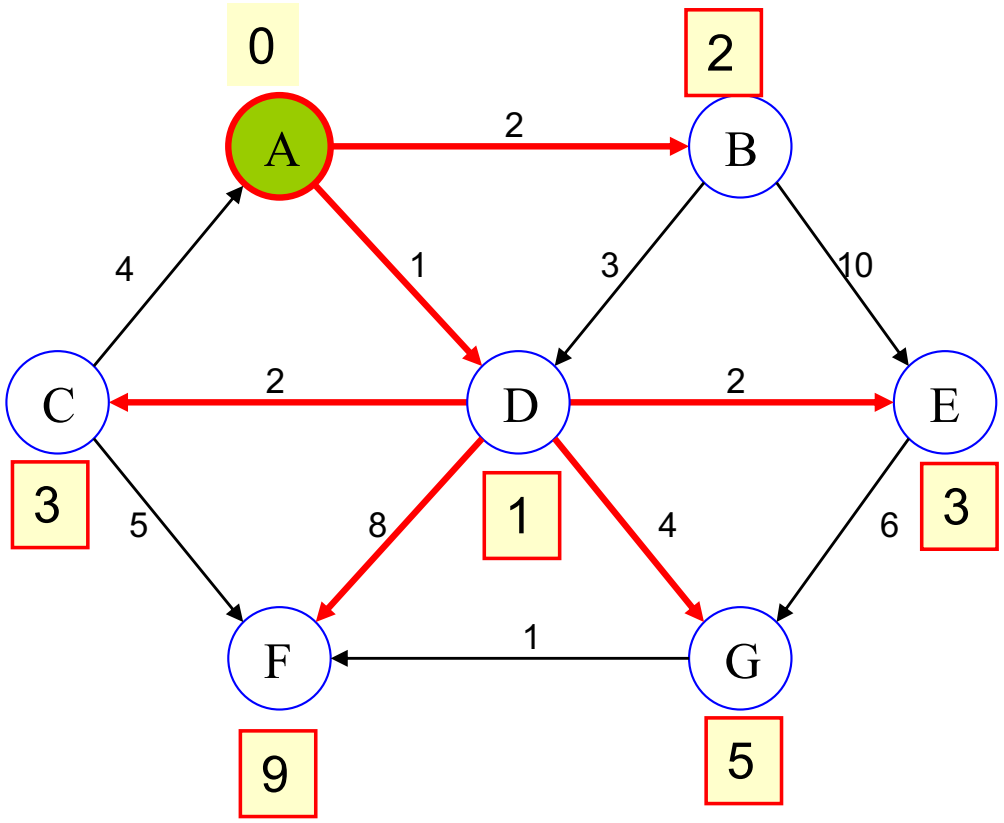

| V | distTo[] | edgeTo |
|---|---|---|
| A | 0 | -- |
| B | ∞ | |
| C | ∞ | |
| D | ∞ | |
| E | ∞ | |
| F | ∞ | |

# Update A's neighbors



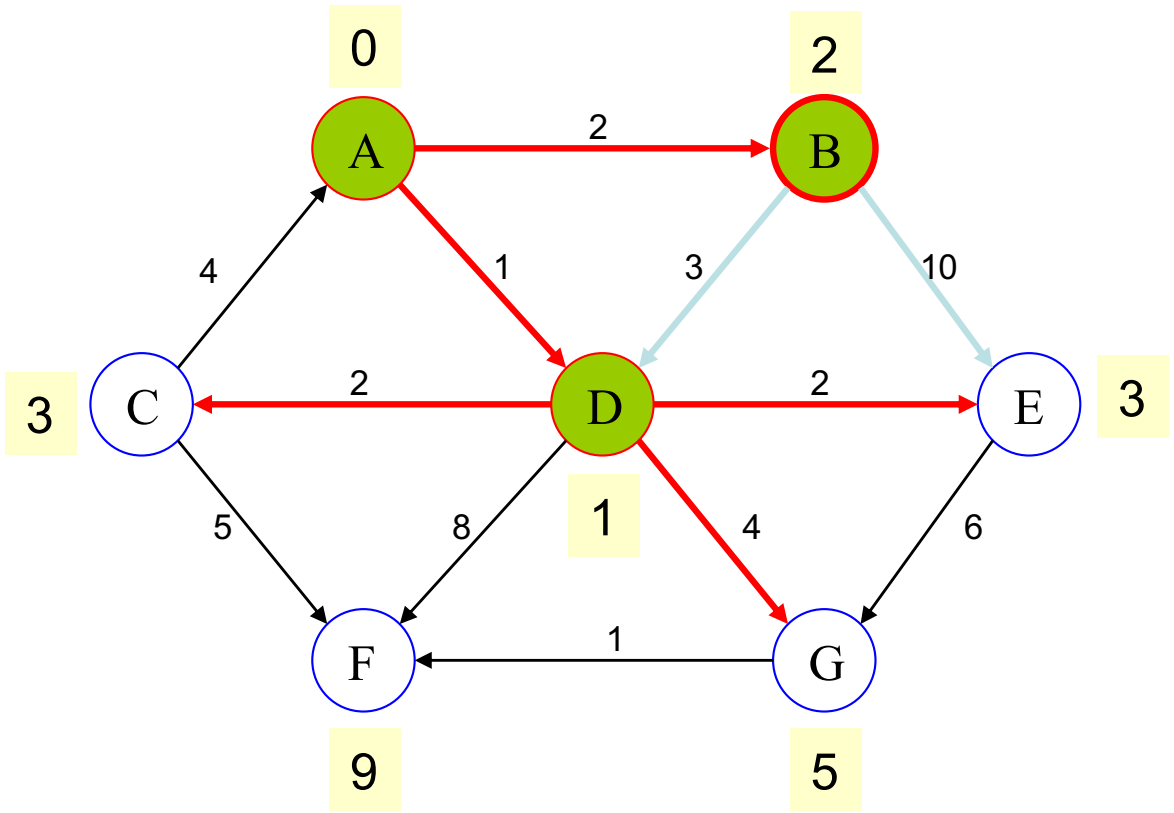

| V | distTo[] | edgeTo |
|---|---|---|
| A | 0 | -- |
| B | **2** | 0 |
| C | ∞ | |
| D | **1** | A |
| E | ∞ | |
| F | ∞ | |

# Update D's neighbors




| V | distTo[] | edgeTo |
|---|---|---|
| **A** | 0 | -- |
| B | **2** | A |
| C | **3** | D |
| **D** | **1** | A |
| E | **3** | D |
| F | **9** | D |
| G | **5** | D |

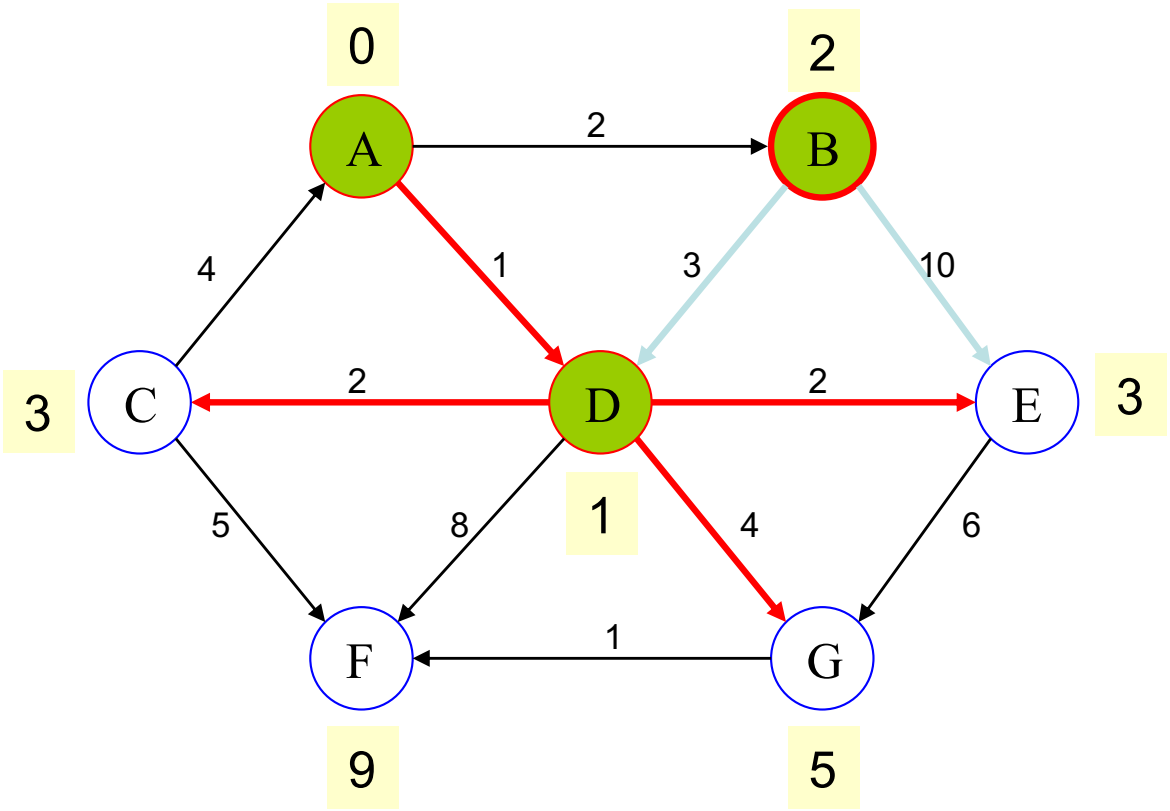
# Update B's neighbors




| V | distTo[] | edgeTo |
|---|---|---|
| **A** | 0 | -- |
| B | **2** | A |
| C | **3** | D |
| **D** | **1** | A |
| E | **3** | D |
| F | **9** | D |
| G | **5** | D |

**No Update**

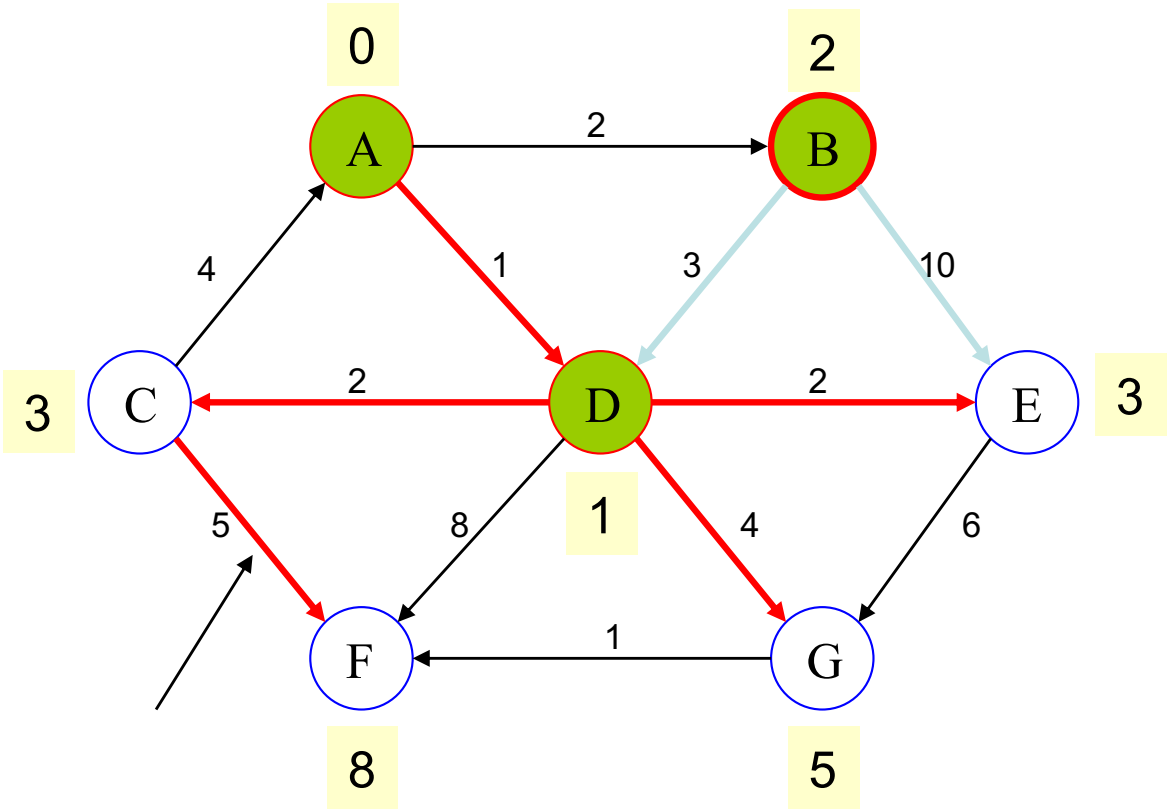
# Update E's neighbors




| V | distTo[] | edgeTo |
|---|---|---|
| **A** | 0 | -- |
| B | **2** | A |
| C | **3** | D |
| **D** | **1** | A |
| E | **3** | D |
| F | **9** | D |
| G | **5** | D |

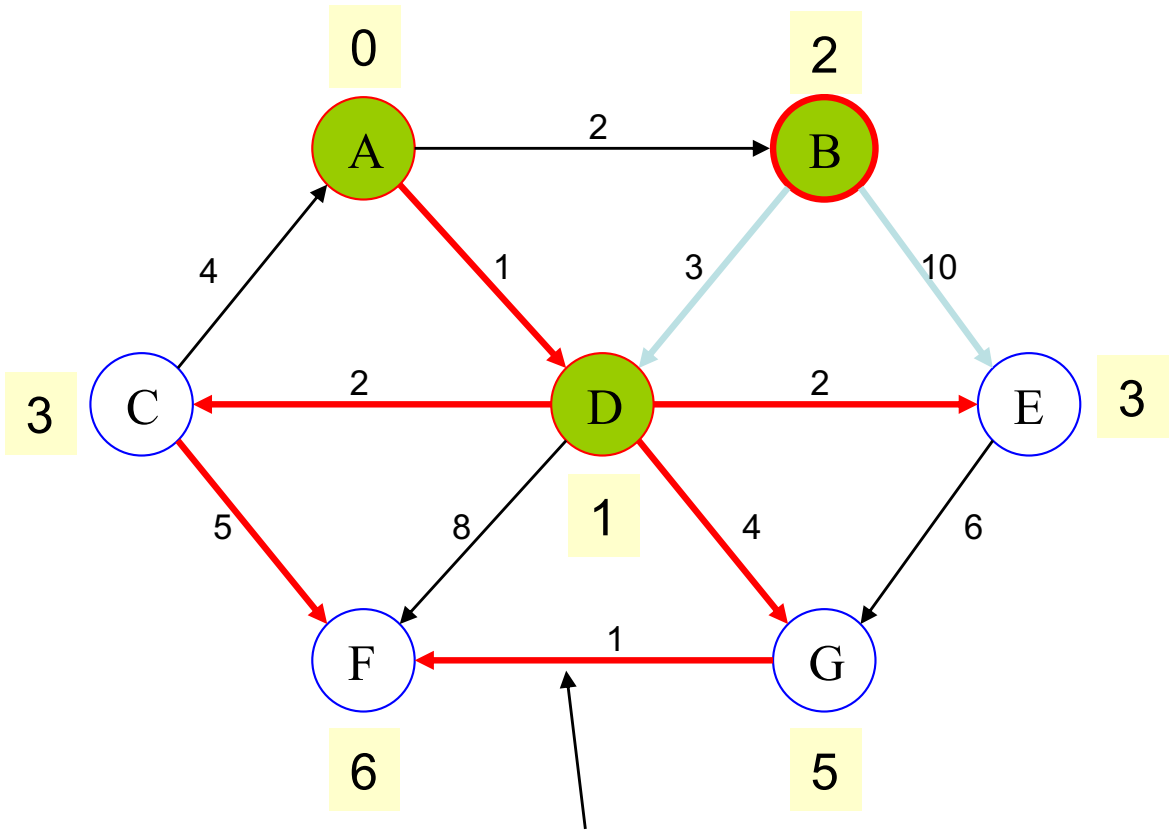
**No Update**

# Update C's neighbors



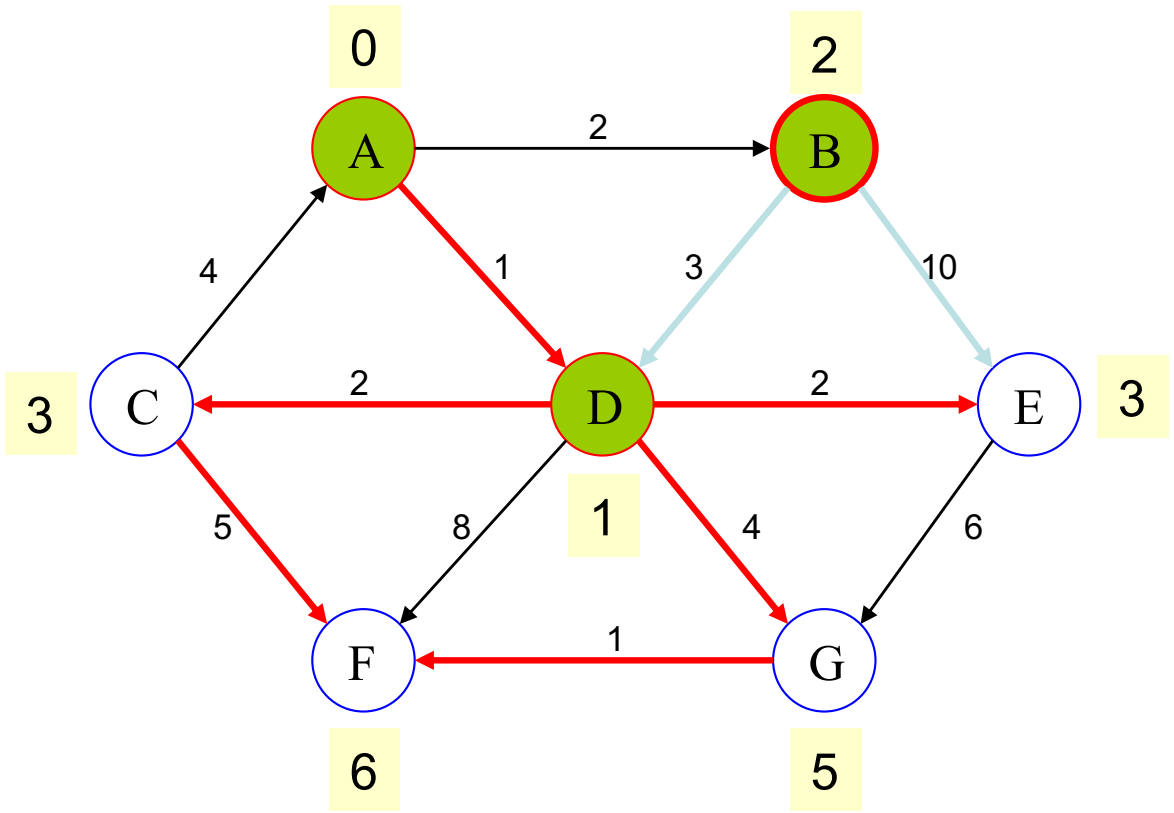

| V | distTo[] | edgeTo |
|---|---|---|
| **A** | 0 | -- |
| B | **2** | A |
| C | **3** | D |
| **D** | **1** | A |
| **E** | **3** | D |
| F | **8** | C |
| G | **5** | D |

# Update G's neighbors




| V | distTo[] | edgeTo |
|---|---|---|
| **A** | 0 | -- |
| B | **2** | A |
| C | **3** | D |
| **D** | **1** | A |
| **E** | **3** | D |
| F | **6** | G |
| G | **5** | D |

# Update F's neighbors

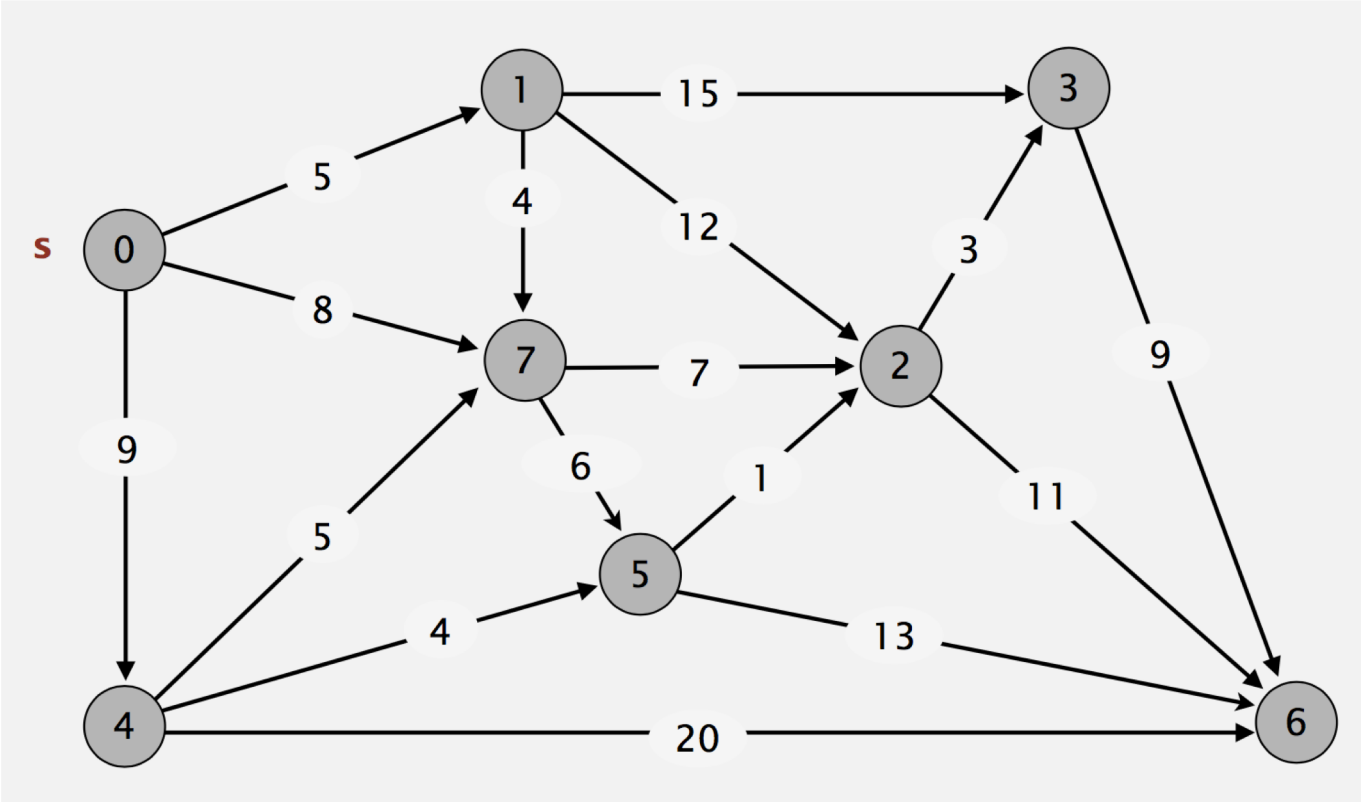



| V | distTo[] | edgeTo |
|---|---|---|
| **A** | 0 | -- |
| B | **2** | A |
| C | **3** | D |
| **D** | **1** | A |
| E | **3** | D |
| F | **6** | G |
| G | **5** | D |

**No Update**

# Dijkstra's algorithm Demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges pointing from that vertex.



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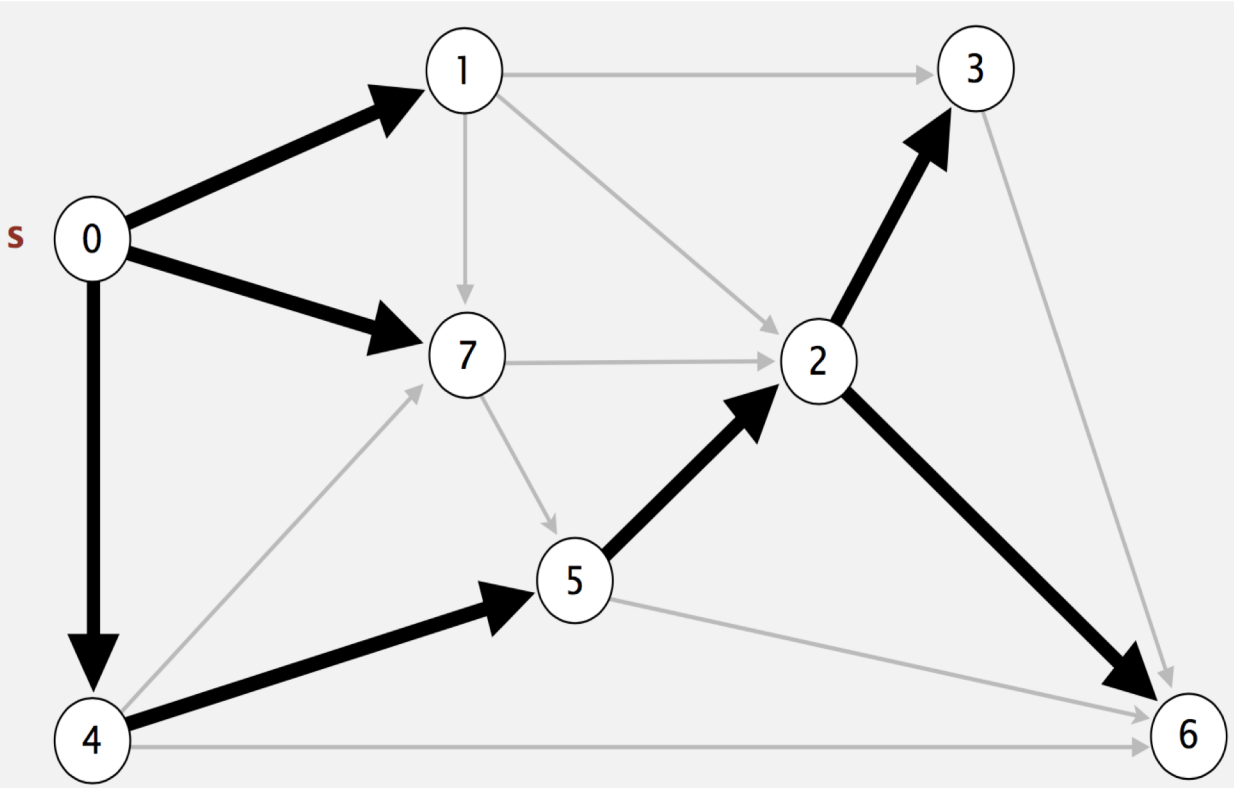


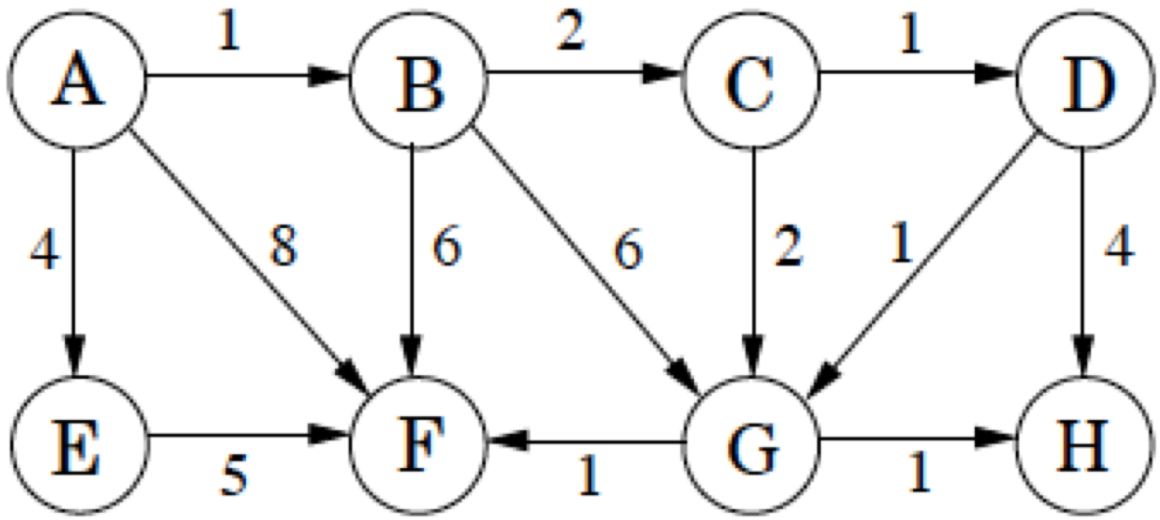

| v | distTo[] | edgeTo[] |
|---|---|---|
| 0 | 0.0 | - |
| 1 | 5.0 | 0→1 |
| 2 | 14.0 | 5→2 |
| 3 | 17.0 | 2→3 |
| 4 | 9.0 | 0→4 |
| 5 | 13.0 | 4→5 |
| 6 | 25.0 | 2→6 |
| 7 | 8.0 | 0→7 |

# Dijkstra's algorithm Implementation

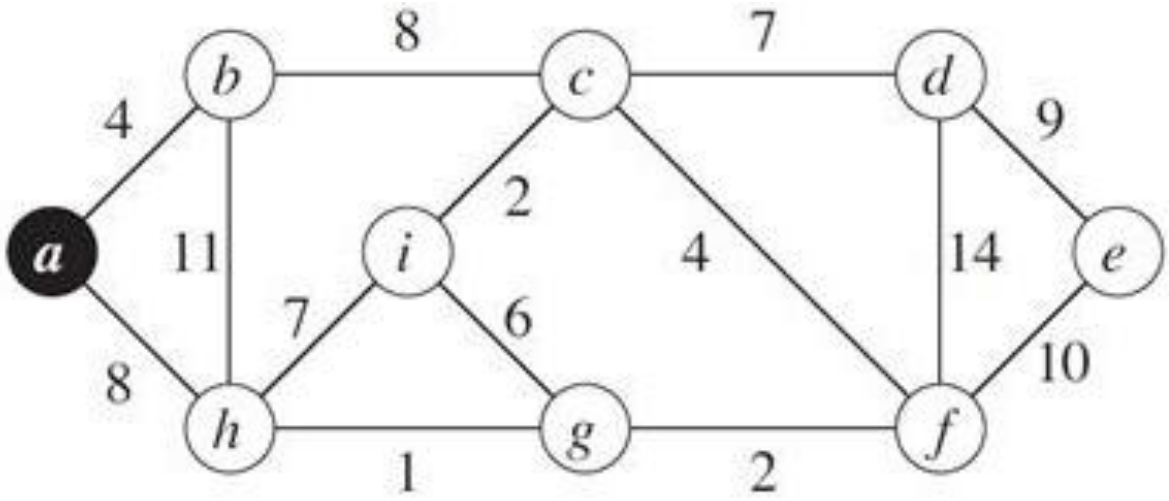
```
public class DijkstraSP{
   private DirectedEdge[] edgeTo;
   private double[] distTo;
   private IndexMinPQ<Double> pq;

  public DijkstraSP(EdgeWeightedDigraph G, int s) {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];
      pq = new IndexMinPQ<Double>(G.V());
      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;
      pq.insert(s, 0.0);
      while (!pq.isEmpty()){
         int v = pq.delMin();
         for (DirectedEdge e : G.adj(v))
             relax(e);
      }
  }
}
```

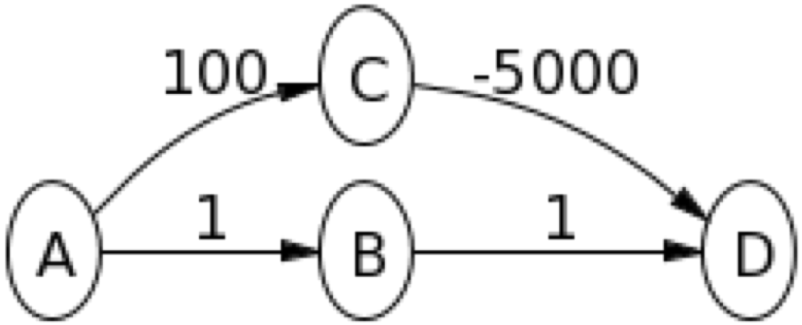
# Shortest Path Demo



# Shortest Path Demo



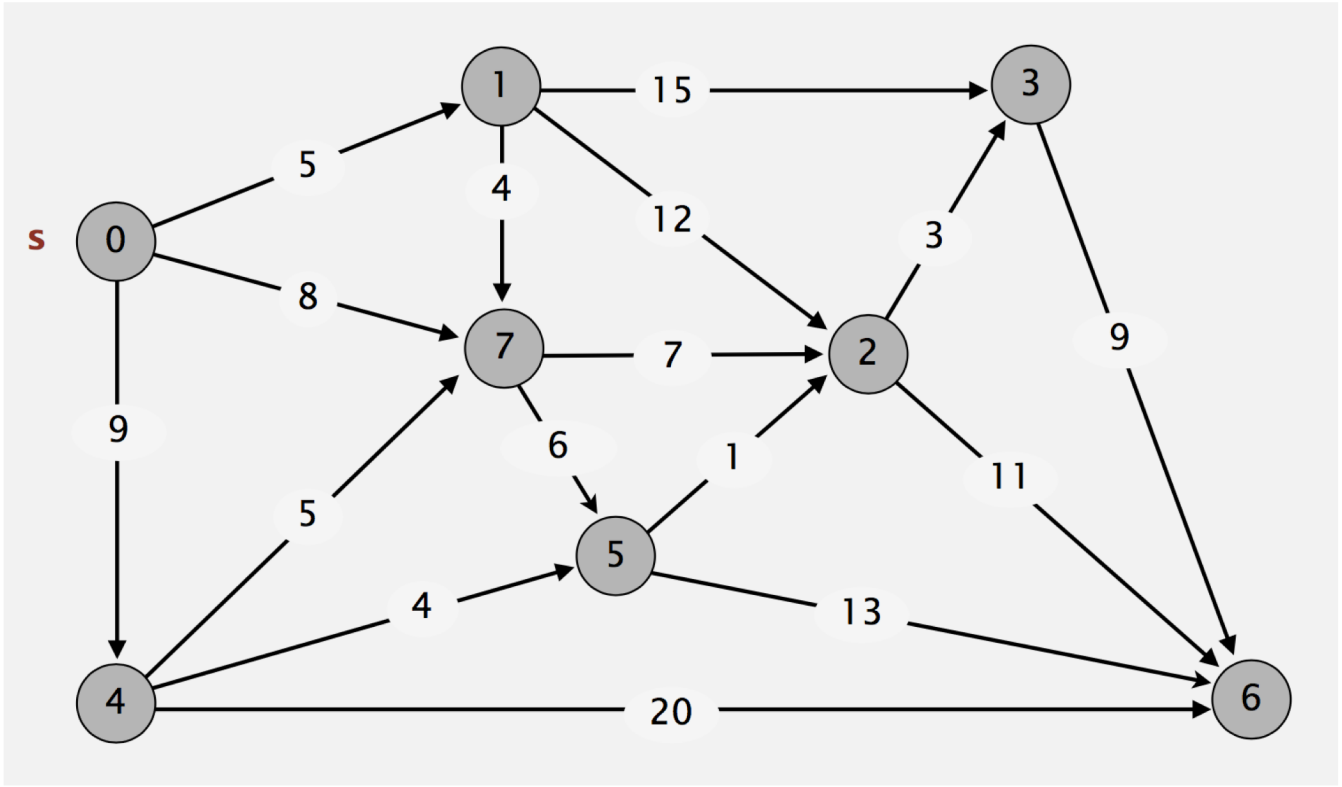
# Shortest Path Demo



# Acyclic shortest paths

- Consider vertices in topological order. Relax all edges pointing from that vertex.

0 1 4 7 5 2 3 6



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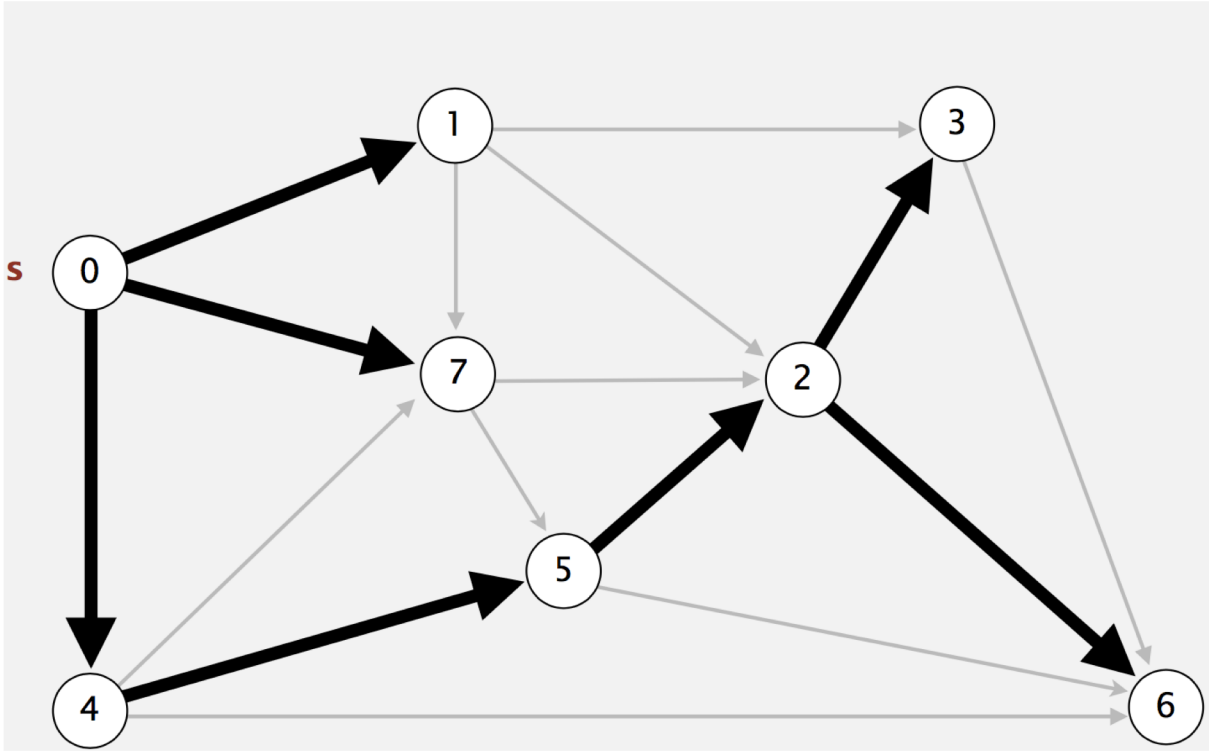
- Consider vertices in topological order.
- Relax all edges pointing from that vertex.




0 1 4 7 5 2 3 6

| v | distTo[] | edgeTo[] |
|---|---|---|
| 0 | 0.0 | - |
| 1 | 5.0 | 0→1 |
| 2 | 14.0 | 5→2 |
| 3 | 17.0 | 2→3 |
| 4 | 9.0 | 0→4 |
| 5 | 13.0 | 4→5 |
| 6 | 25.0 | 2→6 |
| 7 | 8.0 | 0→7 |

# Longest paths in edge-weighted DAGs

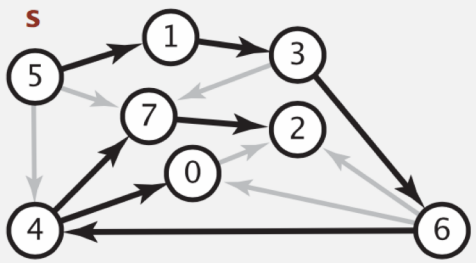
- Formulate as a shortest paths problem in edge-weighted DAGs.
  - Negate all weights.
  - Find shortest paths.
  - Negate weights in result
- Key point. Topological sort algorithm works even with negative weights.

**longest paths input**

```
5->4  0.35
4->7  0.37
5->7  0.28
5->1  0.32
4->0  0.38
0->2  0.26
3->7  0.39
1->3  0.29
7->2  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```

**shortest paths input**

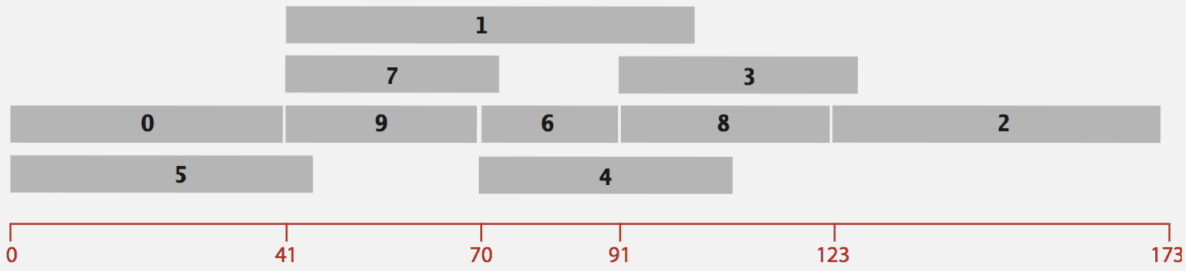
```
5->4 -0.35
4->7 -0.37
5->7 -0.28
5->1 -0.32
4->0 -0.38
0->2 -0.26
3->7 -0.39
1->3 -0.29
7->2 -0.34
6->2 -0.40
3->6 -0.52
6->0 -0.58
6->4 -0.93
```



# Longest paths in edge-weighted DAGs

- Parallel job scheduling.
  - Given a set of jobs with durations and precedence constraints, schedule the jobs (by finding a start time for each) so as to achieve the minimum completion time, while respecting the constraints.

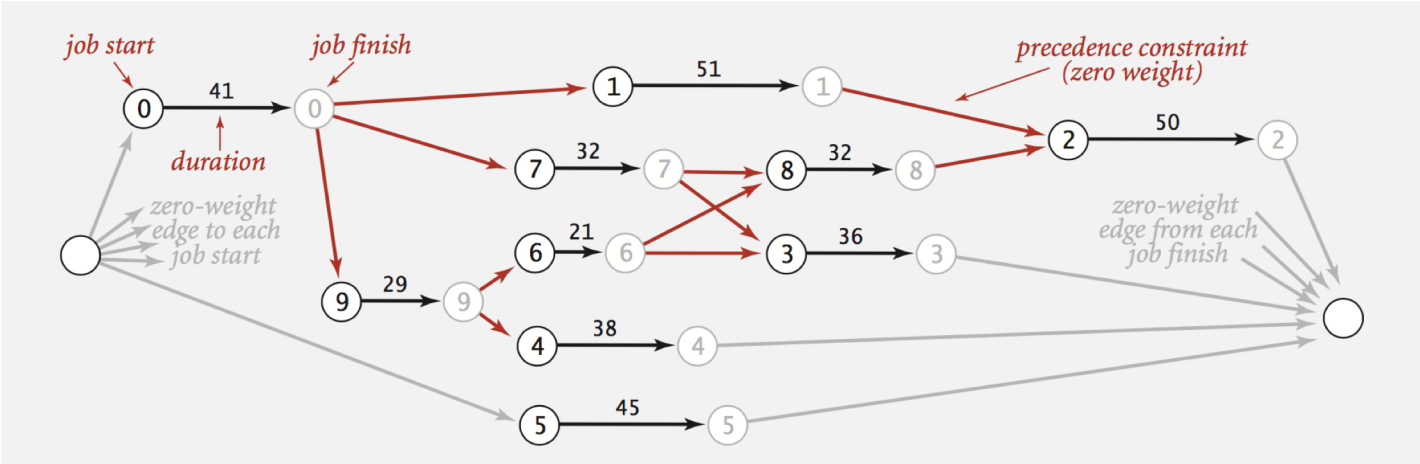
| job | duration | must complete before |
|---|---|---|
| 0 | 41.0 | 1 7 9 |
| 1 | 51.0 | 2 |
| 2 | 50.0 | |
| 3 | 36.0 | |
| 4 | 38.0 | |
| 5 | 45.0 | |
| 6 | 21.0 | 3 8 |
| 7 | 32.0 | 3 8 |
| 8 | 32.0 | 2 |
| 9 | 29.0 | 4 6 |




Parallel job scheduling solution

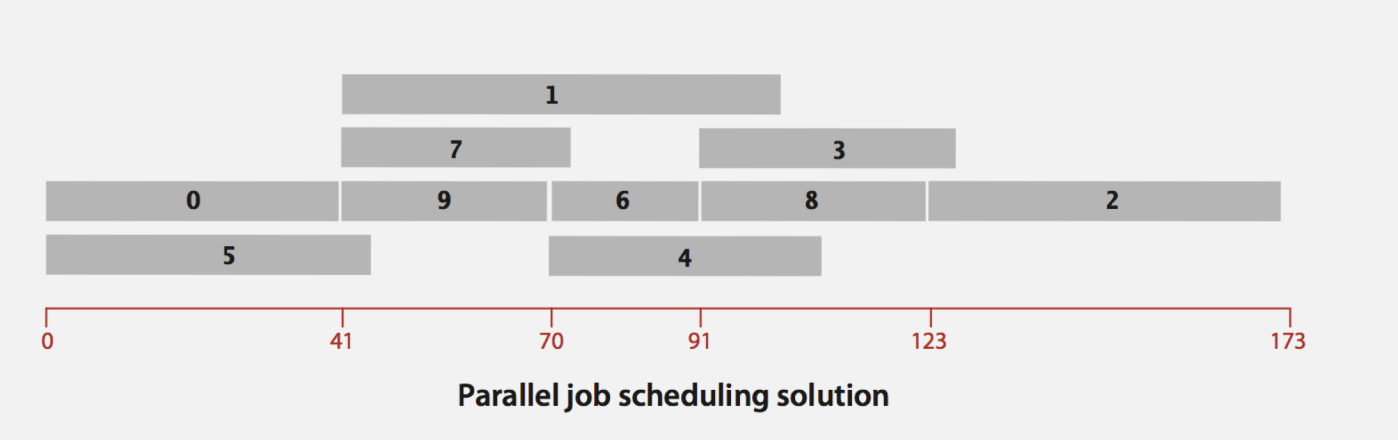
# Critical path method

- To solve a parallel job-scheduling problem, create edge-weighted DAG:
  - Source and sink vertices.
  - Two vertices (begin and end) for each job.
  - Three edges for each job.
    - Begin to end (weighted by duration)
    - Source to begin(0 weight)
    - End to sink(0 weight)
- One edge for each precedence constraint (0 weight).

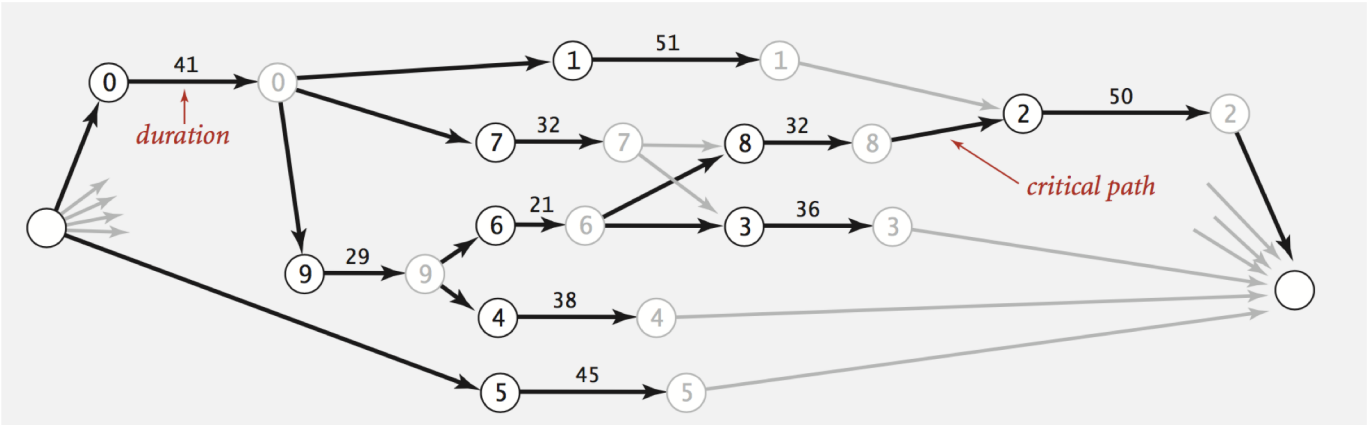


# Critical path method

Use longest path from the source to schedule each job.

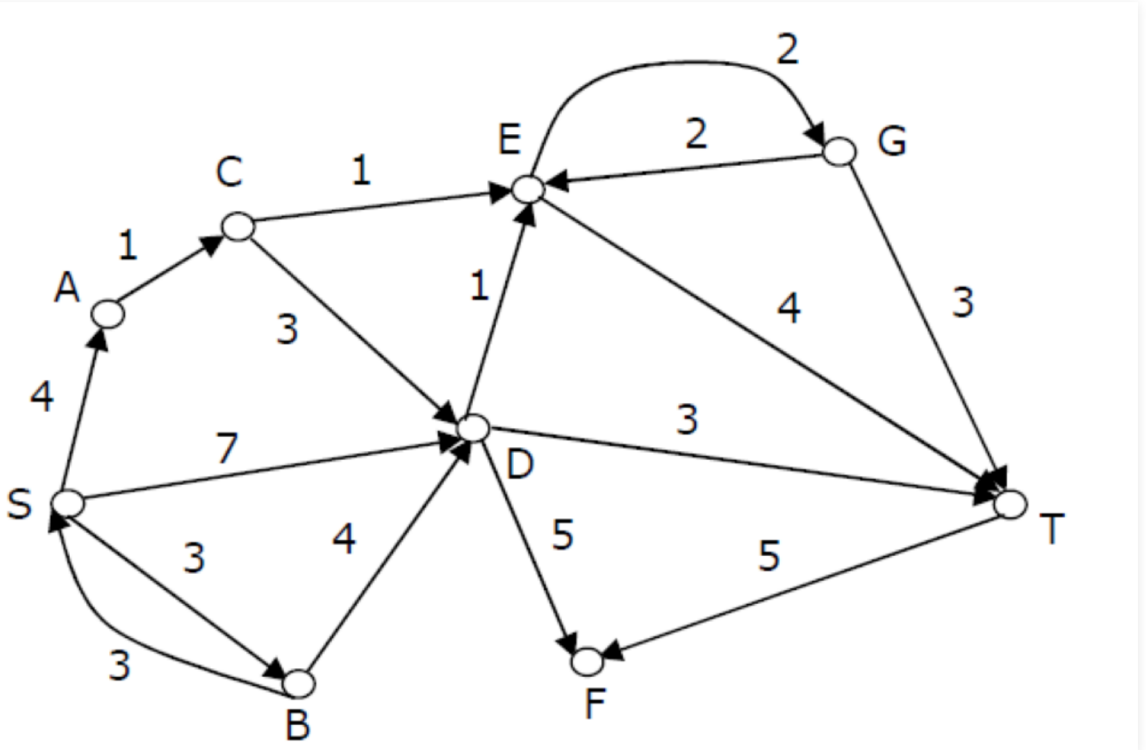



Parallel job scheduling solution



# Quiz 1

There are multiple shortest paths between vertices S and T. Which one will be reported by Dijstra's shortest path algorithm?




A. SDT
B. SBDT
C. SACDT
D. SACET

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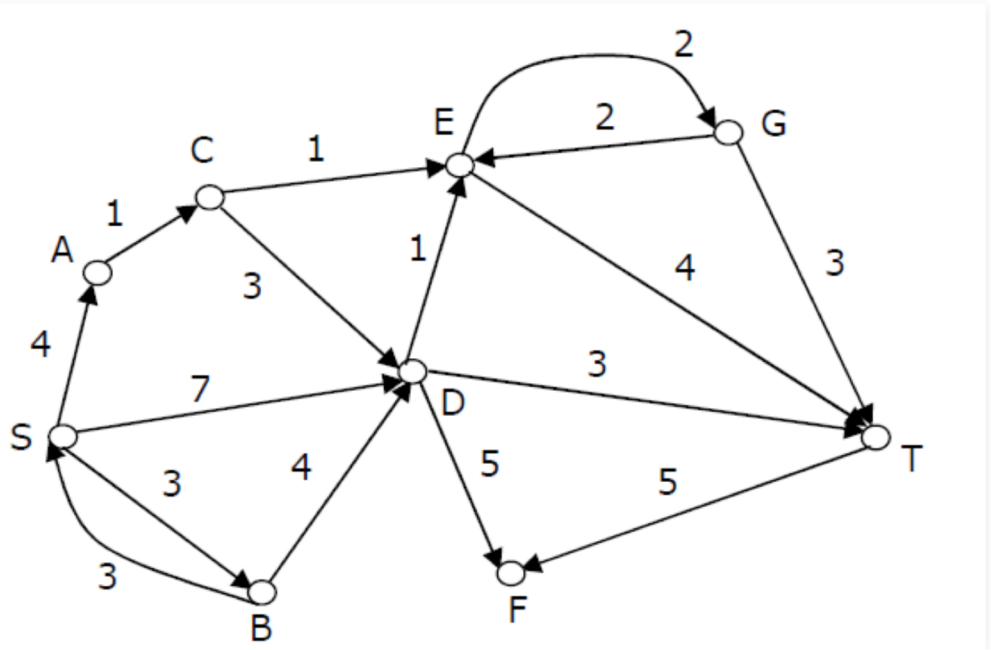


A. SDT
B. SBDT
C. SACDT
D. SACET

# Quiz 2

In an unweighted, undirected connected graph, the shortest path from a node S to every other node is computed most efficiently, in terms of time complexity by

A. Dijkstra's algorithm starting from S.
B. Performing a DFS starting from S.
C. Performing a BFS starting from S.
D. None of the above

# Quiz 2

In an unweighted, undirected connected graph, the shortest path from a node S to every other node is computed most efficiently, in terms of time complexity by

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C. Performing a BFS starting from S.
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