CMSC 330: Organization of Programming Languages

DFAs, and NFAs, and Regexps

The story so far, and what's next

- Goal: Develop an algorithm that determines whether a string *s* is matched by regex *R*
  - I.e., whether *s* is a member of *R*'s *language*

- Approach: Convert *R* to a finite automaton *FA* and see whether *s* is accepted by *FA*
  - Details: Convert *R* to a *nondeterministic FA* (NFA), which we then convert to a *deterministic FA* (DFA),
    - which enjoys a fast acceptance algorithm

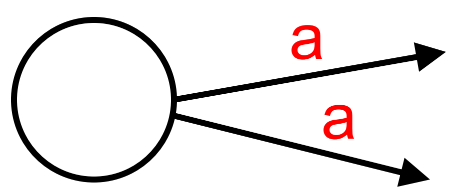
Two Types of Finite Automata

- Deterministic Finite Automata (DFA)
  - Exactly one sequence of steps for each string
    - Easy to implement acceptance check
  - All examples so far

- Nondeterministic Finite Automata (NFA)
  - May have many sequences of steps for each string
  - Accepts if any path ends in final state at end of string
  - More compact than DFA
    - But more expensive to test whether a string matches

Comparing DFAs and NFAs

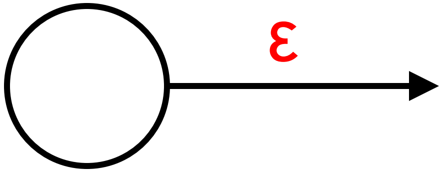
- NFAs can have more than one transition leaving a state on the same symbol



- DFAs allow only one transition per symbol
  - I.e., transition function must be a valid function
  - DFA is a special case of NFA

Comparing DFAs and NFAs (cont.)

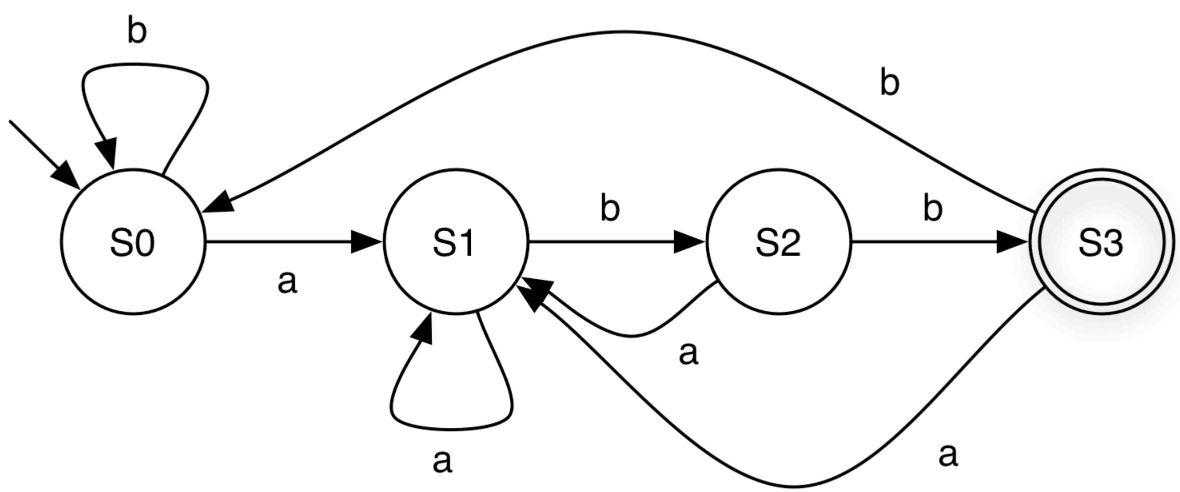
- NFAs may have transitions with empty string label
  - May move to new state without consuming character



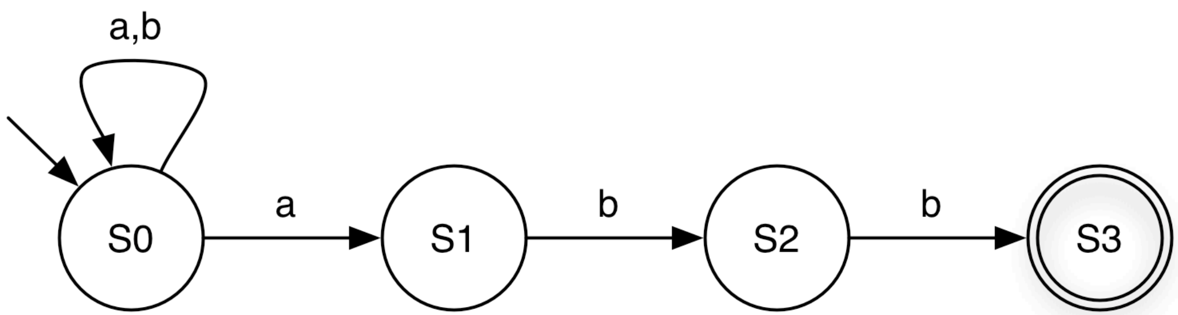
ε-transition

- DFA transition must be labeled with symbol
  - DFA is a special case of NFA

DFA for (a|b)*abb

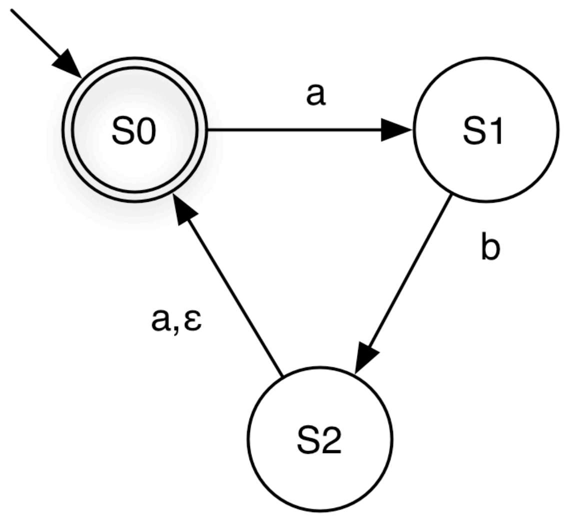


NFA for (a|b)*abb



- ba
  - Has paths to either S0 or S1
  - Neither is final, so rejected
- babaabb
  - Has paths to different states
  - One path leads to S3, so accepts string

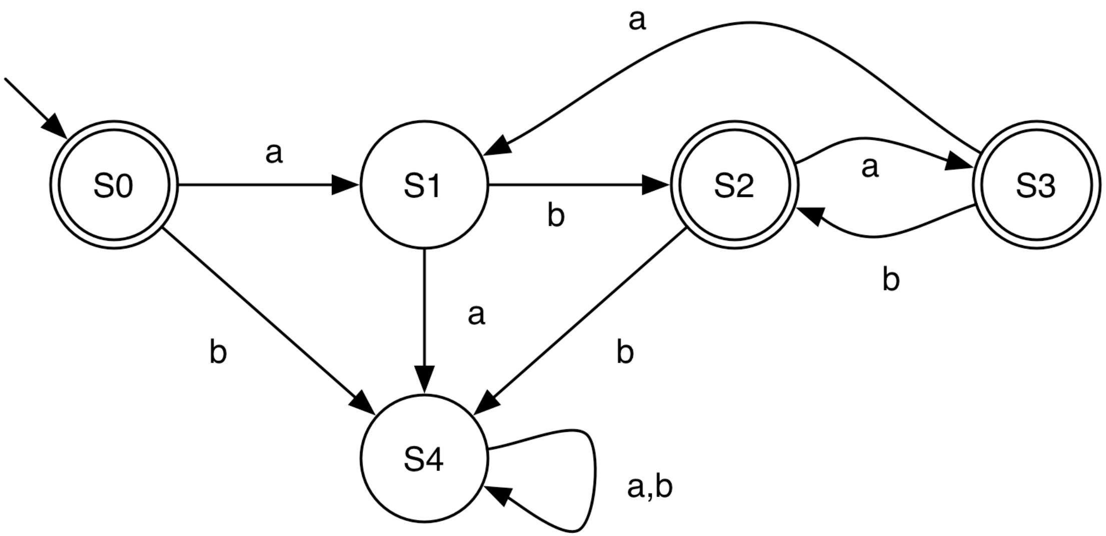
NFA for (ab|aba)*

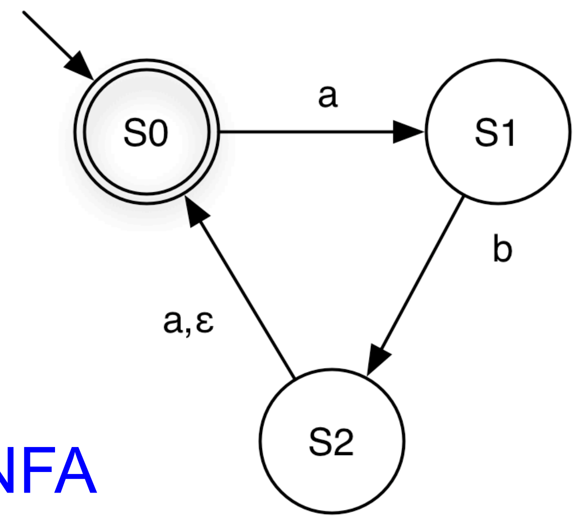


- aba
  - Has paths to states S0, S1
- ababa
  - Has paths to S0, S1
  - Need to use ε-transition

Comparing NFA and DFA for (ab|aba)*

DFA



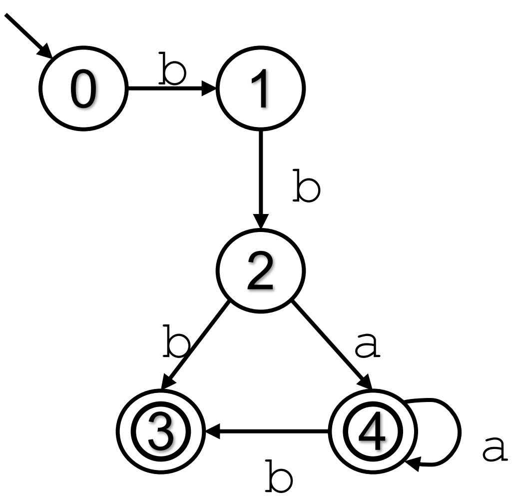


NFA

Quiz 1: Which DFA matches this regexp?

`b(b|a+b?)`

A.



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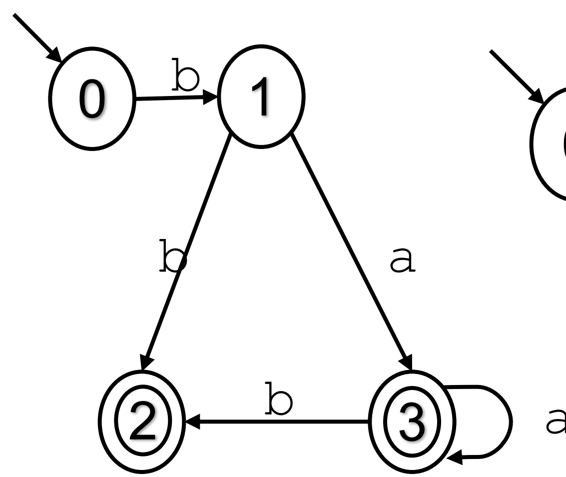

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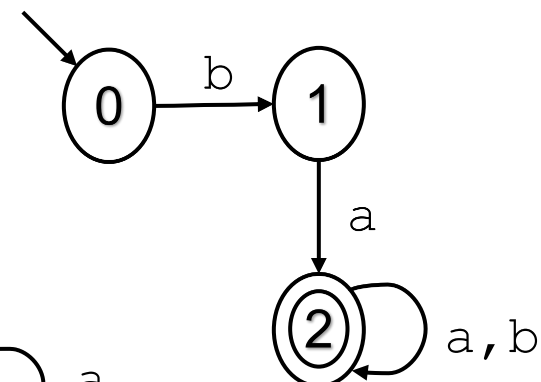

D. None of the above

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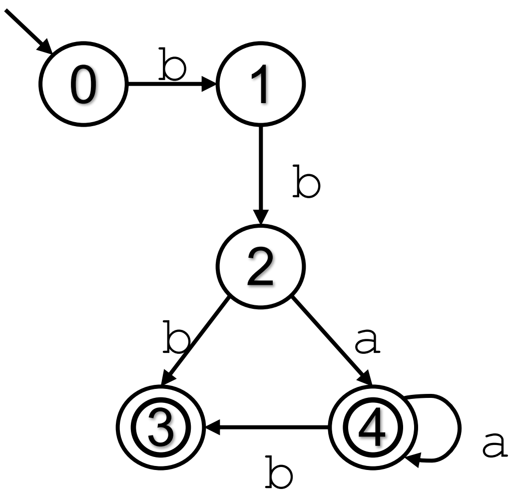

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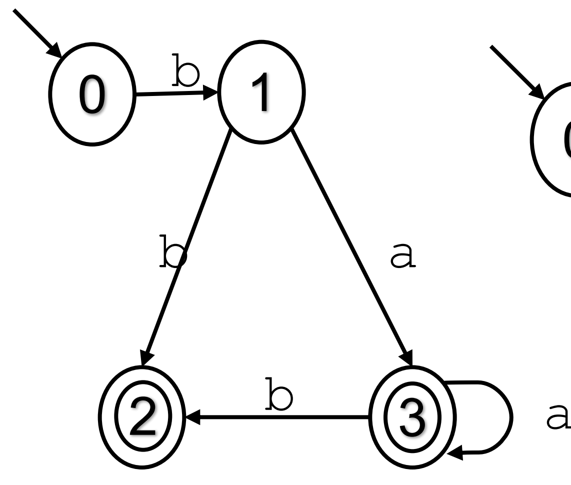
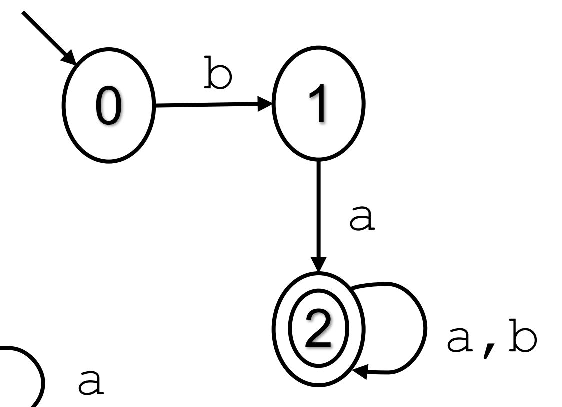


C.



D. None of the above

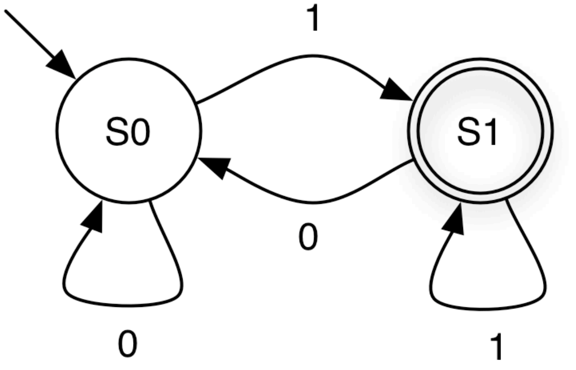
Formal Definition

- A deterministic finite automaton *(DFA)* is a 5-tuple $(\Sigma, Q, q_0, F, \delta)$ where
  - Σ is an alphabet
  - Q is a nonempty set of states
  - $q_0 \in Q$ is the start state
  - $F \subseteq Q$ is the set of final states
  - $\delta : Q \times \Sigma \rightarrow Q$ specifies the DFA's transitions
    - What's this definition saying that δ is?
- A DFA accepts s if it stops at a final state on s

Formal Definition: Example

- $\Sigma = \{0, 1\}$
- $Q = \{S0, S1\}$
- $q_0 = S0$
- $F = \{S1\}$
-

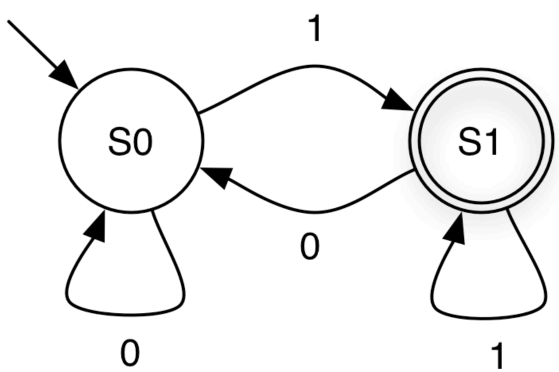
		symbol	
	δ	0	1
input state	S0	S0	S1
	S1	S0	S1



or as { (S0,0,S0),(S0,1,S1),(S1,0,S0),(S1,1,S1) }

Implementing DFAs (one-off)

It's easy to build a program which mimics a DFA




```
cur_state = 0;
while (1) {

  symbol = getchar();

  switch (cur_state) {

    case 0: switch (symbol) {
              case '0':  cur_state = 0; break;
              case '1':  cur_state = 1; break;
              case '\n': printf("rejected\n"); return 0;
              default:   printf("rejected\n"); return 0;
            }
            break;

    case 1: switch (symbol) {
              case '0':  cur_state = 0; break;
              case '1':  cur_state = 1; break;
              case '\n': printf("accepted\n"); return 1;
              default:   printf("rejected\n"); return 0;
            }
            break;

    default: printf("unknown state; I'm confused\n");
             break;
  }
}
```

Implementing DFAs (generic)

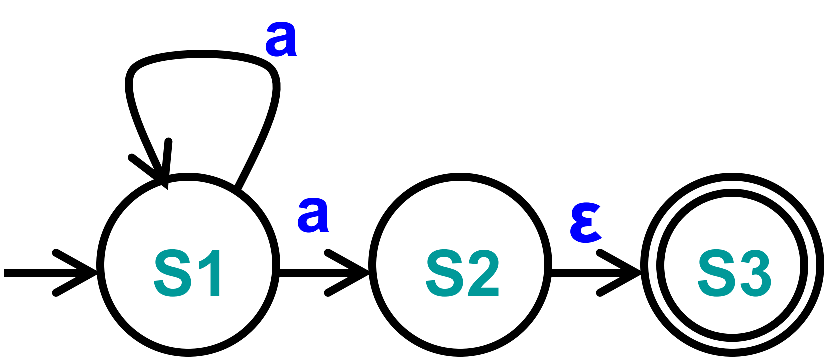
More generally, use generic table-driven DFA

given components (Σ, Q, q_0, F, δ) of a DFA:
let q = q_0
while (there exists another symbol σ of the input string)
    q := δ(q, σ);
if q $\in$ F then
    accept
else reject

- q is just an integer
- Represent δ using arrays or hash tables
- Represent F as a set

Nondeterministic Finite Automata (NFA)

- An *NFA* is a 5-tuple $(\Sigma, Q, q_0, F, \delta)$ where
  - Σ, Q, q0, F as with DFAs
  - $\delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times Q$ specifies the NFA's transitions



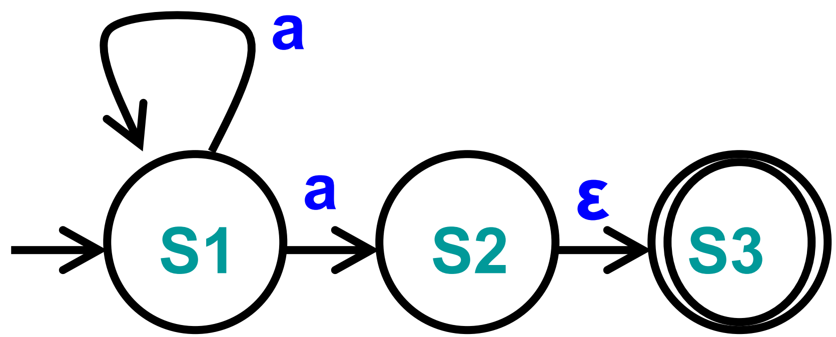
Example

- $\Sigma = \{a\}$
- $Q = \{S1, S2, S3\}$
- $q_0 = S1$
- $F = \{S3\}$
- $\delta = \{ (S1,a,S1), (S1,a,S2), (S2,\varepsilon,S3) \}$

- An NFA accepts s if there is at least one path via s from the NFA's start state to a final state

NFA Acceptance Algorithm (Sketch)

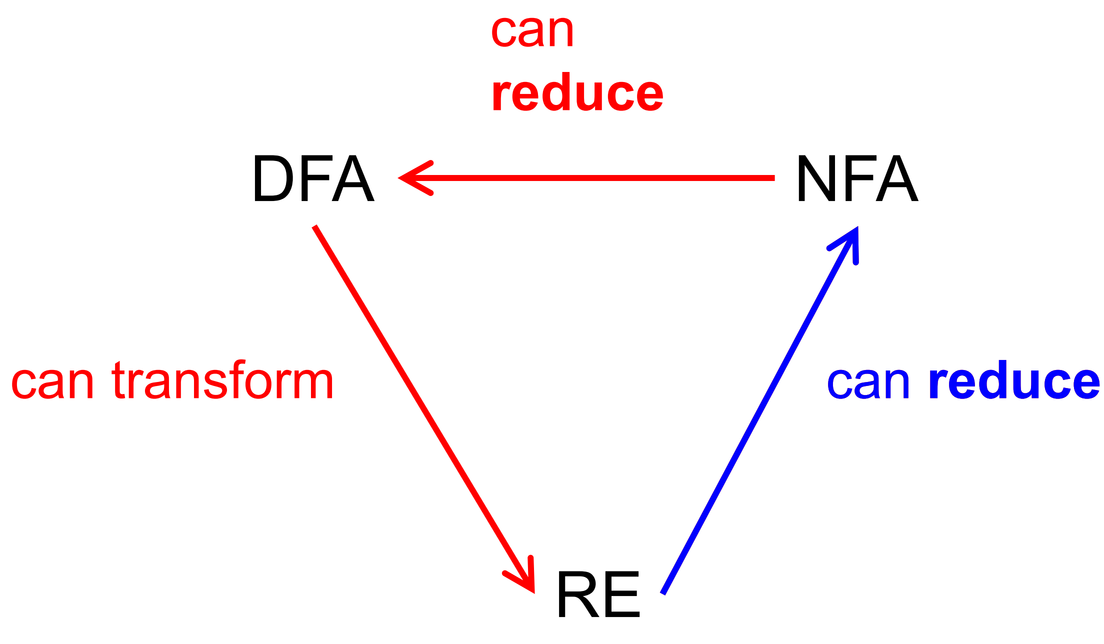
- When NFA processes a string s
  - NFA must keep track of several "current states"
    - Due to multiple transitions with same label, and ε-transitions
  - If any current state is final when done then accept s
- Example
  - After processing "a"
    - NFA may be in states
      S1
      S2
      S3
    - Since S3 is final, s is accepted



- Algorithm is slow, space-inefficient; prefer DFAs!

Relating REs to DFAs and NFAs

- Regular expressions, NFAs, and DFAs accept the same languages! *Can convert between them*



NB. Both *transform* and *reduce* are historical terms; they mean “convert”

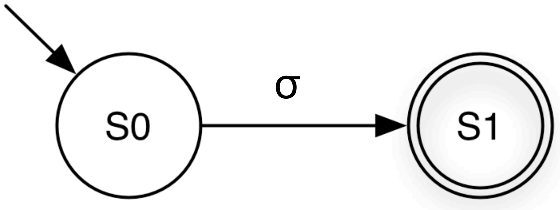
Reducing Regular Expressions to NFAs

- Goal: Given regular expression A, construct NFA: $\langle A \rangle = (\Sigma, Q, q_0, F, \delta)$
  - Remember regular expressions are defined recursively from primitive RE languages
  - Invariant: $|F| = 1$ in our NFAs
    - Recall F = set of final states

- Will define $\langle A \rangle$ for base cases: σ , ε , $\emptyset$
  - Where σ is a symbol in Σ
- And for inductive cases: AB, $A|B$, A^*

Reducing Regular Expressions to NFAs

- Base case: σ

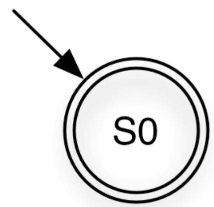


Recall: NFA is $(\Sigma, Q, q_0, F, \delta)$
where
Σ is the alphabet
Q is set of states
q_0 is starting state
F is set of final states
δ is transition relation

<σ> = ({σ}, {S0, S1}, S0, {S1}, {(S0, σ, S1)})

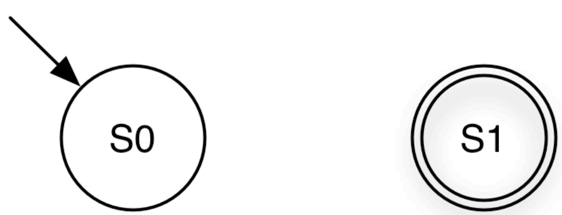
Reduction

- Base case: ε



$<\varepsilon> = (\emptyset, \{S0\}, S0, \{S0\}, \emptyset)$

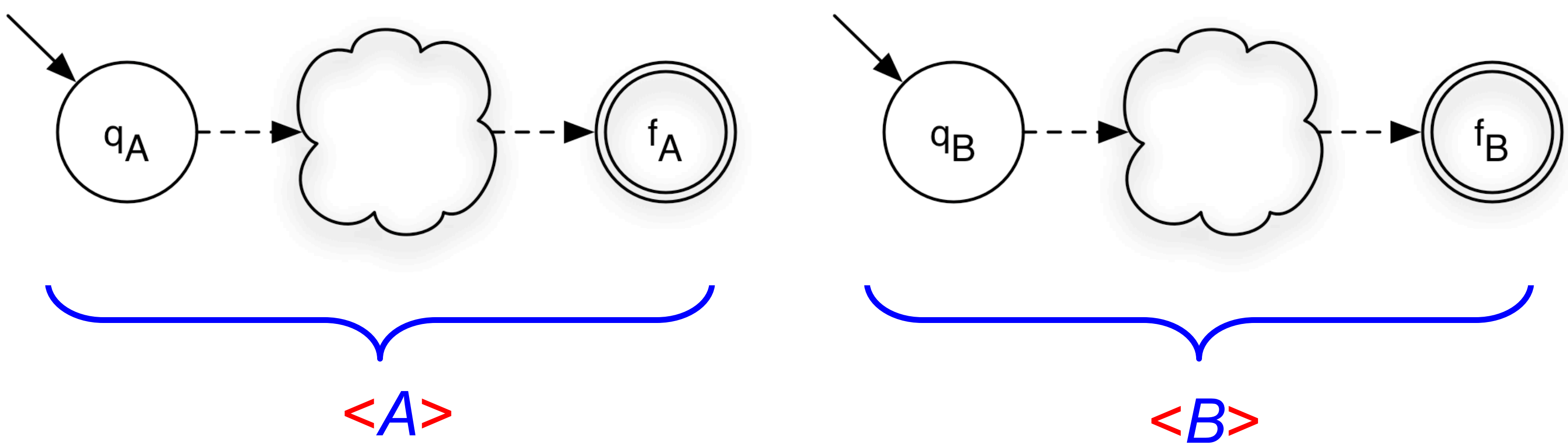
- Base case: ∅



$<\emptyset> = (\emptyset, \{S0, S1\}, S0, \{S1\}, \emptyset)$

Reduction: Concatenation

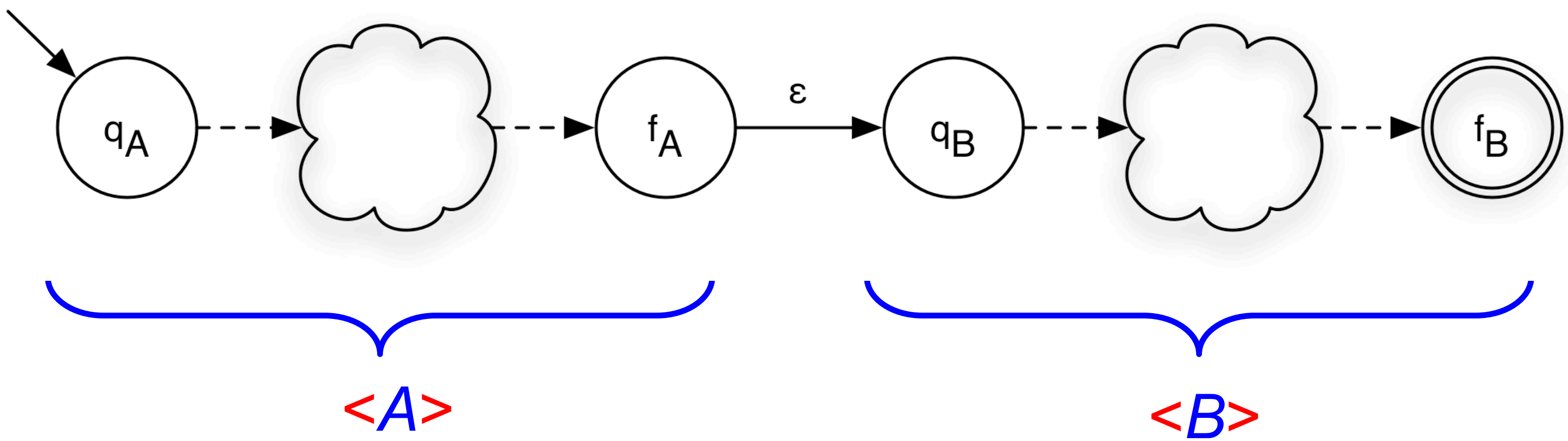
- Induction: *AB*



- <A> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- <B> = $(\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$

Reduction: Concatenation

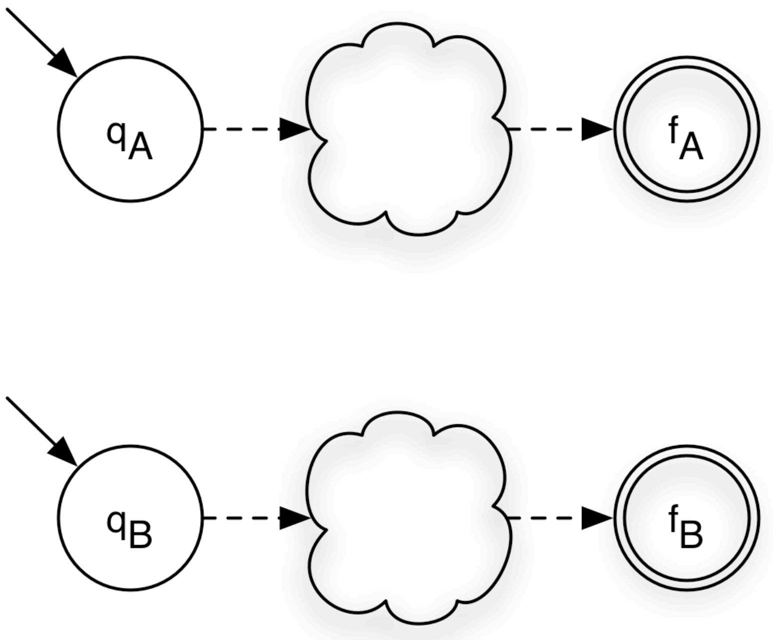
- Induction: *AB*



- $<A> = (\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- $<B> = (\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$
- $<AB> = (\Sigma_A \cup \Sigma_B, Q_A \cup Q_B, q_A, \{f_B\}, \delta_A \cup \delta_B \cup \{(f_A,\varepsilon,q_B)\})$

Reduction: Union

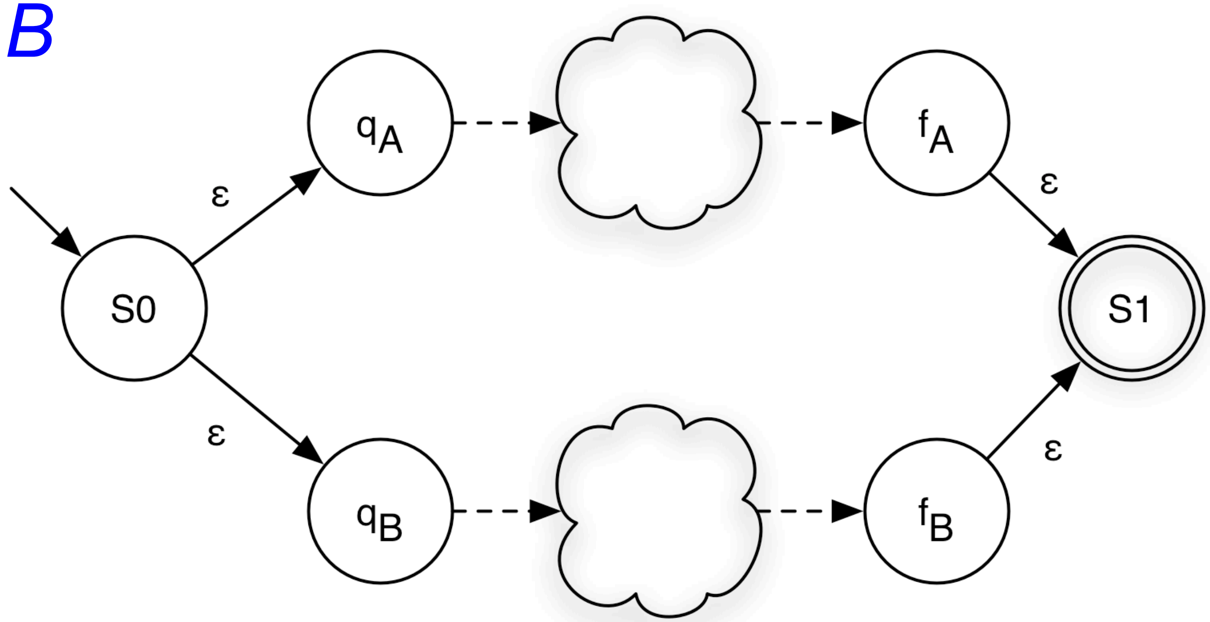
- Induction: *A*|*B*



- <*A*> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- <*B*> = $(\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$

Reduction: Union

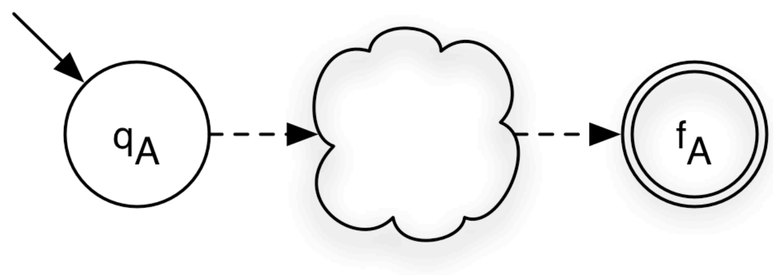
- Induction: *A|B*



- $<A> = (\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- $<B> = (\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$
- $<A|B> = (\Sigma_A \cup \Sigma_B, Q_A \cup Q_B \cup \{S0,S1\}, S0, \{S1\},$
  $\delta_A \cup \delta_B \cup \{(S0,\varepsilon,q_A), (S0,\varepsilon,q_B), (f_A,\varepsilon,S1), (f_B,\varepsilon,S1)\})$

Reduction: Closure

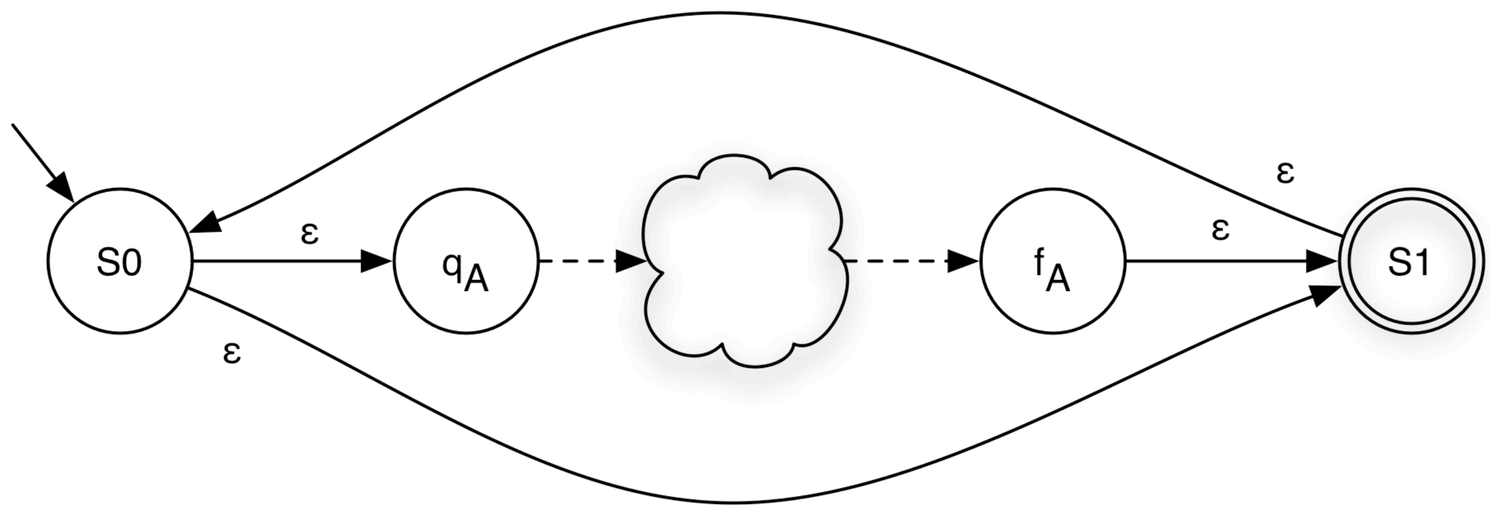
- Induction: *A**



- <*A*> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$

Reduction: Closure

- Induction: *A**

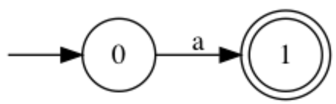


- $<A> = (\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- $<A^*> = (\Sigma_A, Q_A \cup \{S0,S1\}, S0, \{S1\},$

  $\delta_A \cup \{(f_A,\varepsilon,S1), (S0,\varepsilon,q_A), (S0,\varepsilon,S1), (S1,\varepsilon,S0)\})$

Quiz 2: Which NFA matches **a*** ?

A.



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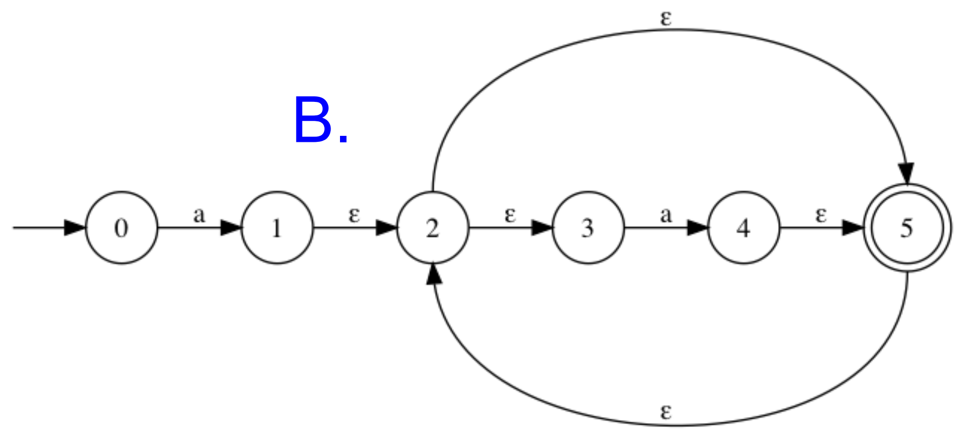

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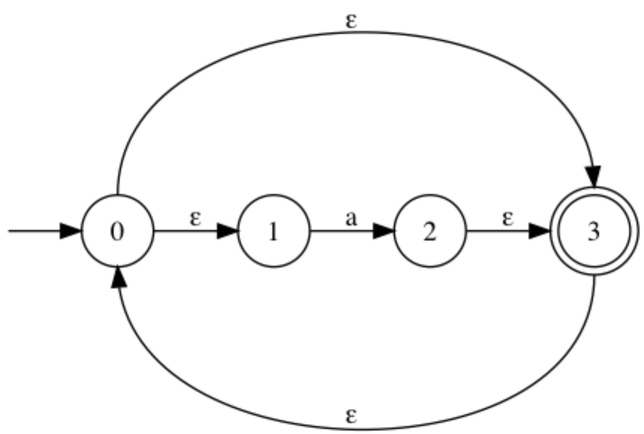

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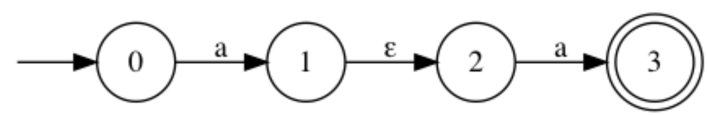

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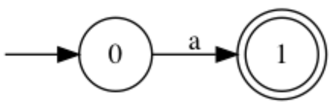

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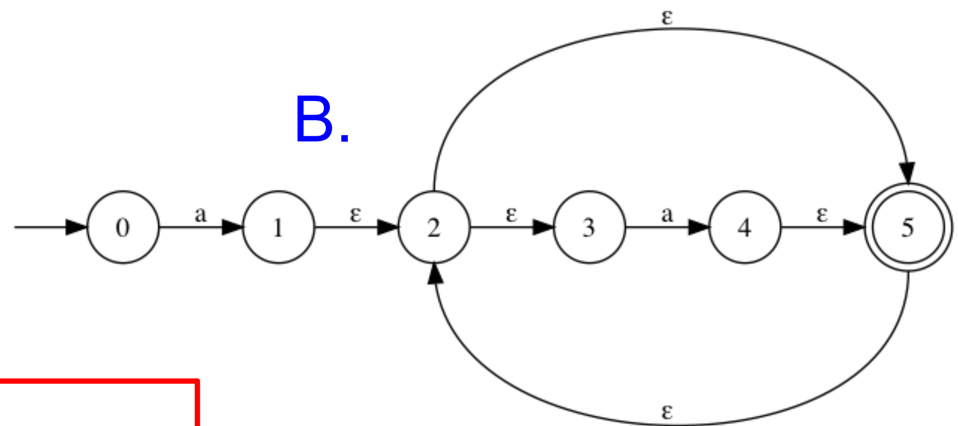

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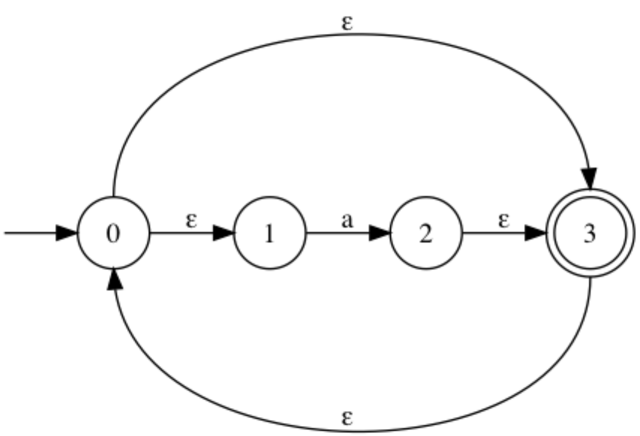

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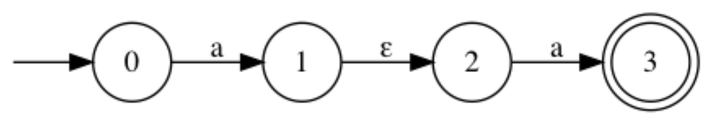

Quiz 3: Which NFA matches **a|b*** ?

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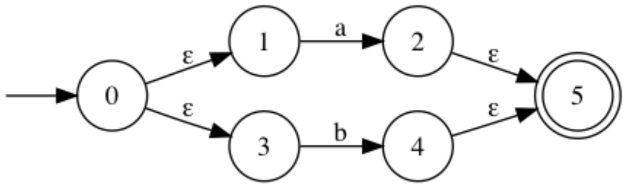

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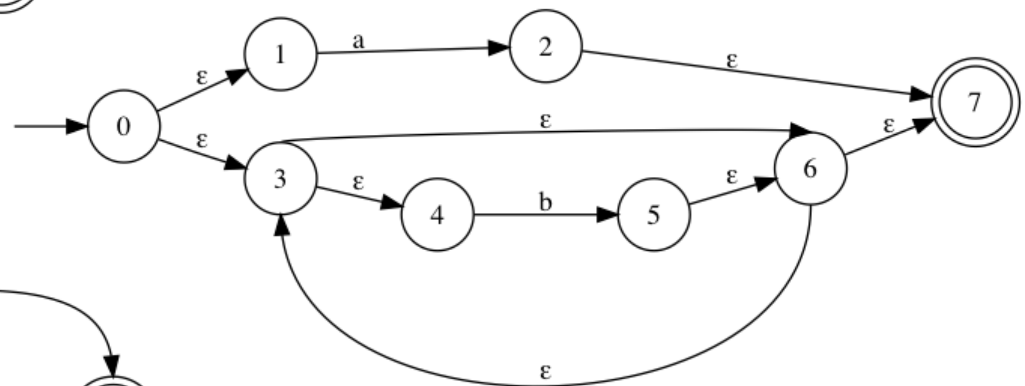

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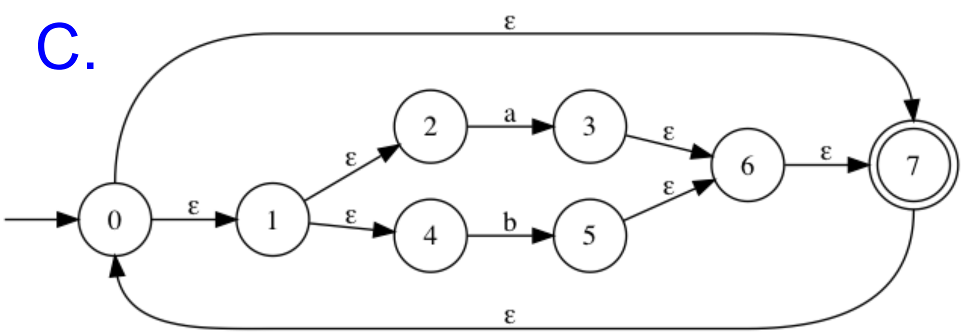

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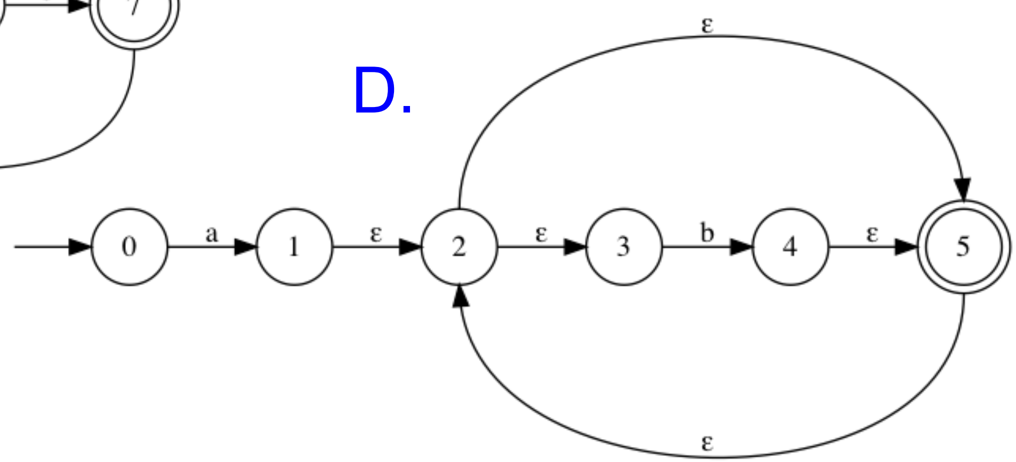

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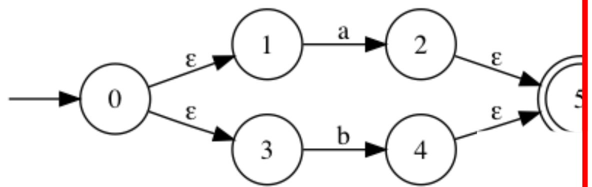

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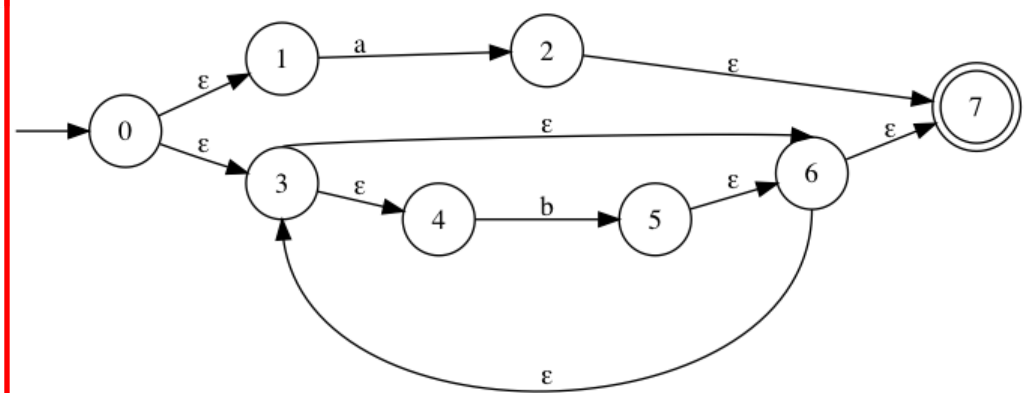

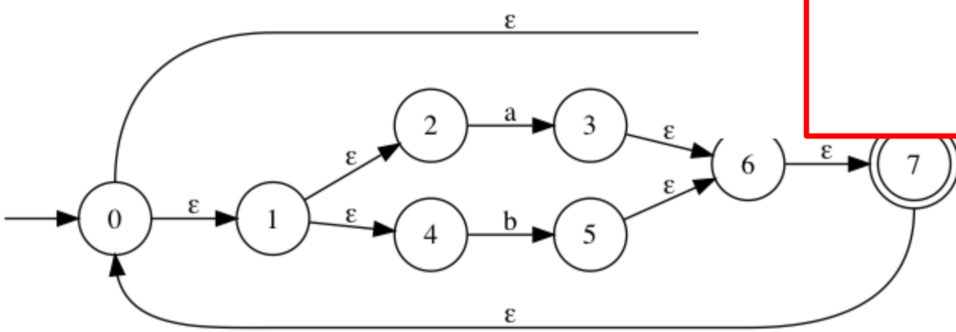

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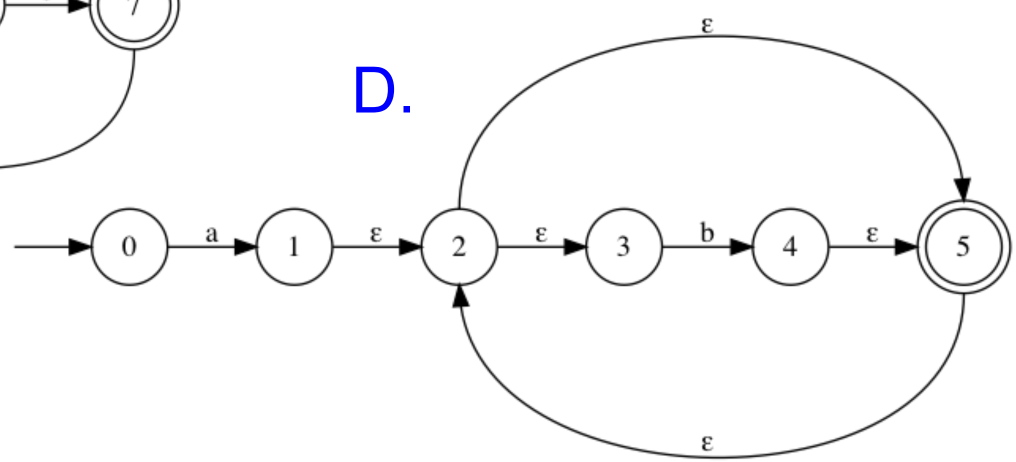


RE → NFA

Draw NFAs for the regular expression (0|1)*110*

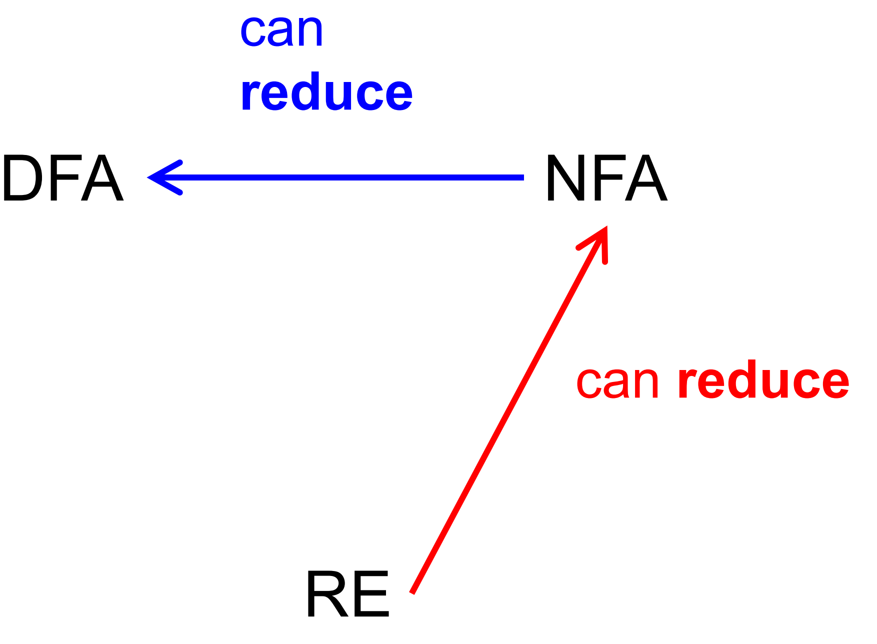
Reduction Complexity

- Given a regular expression *A* of size n...

  Size = # of symbols + # of operations

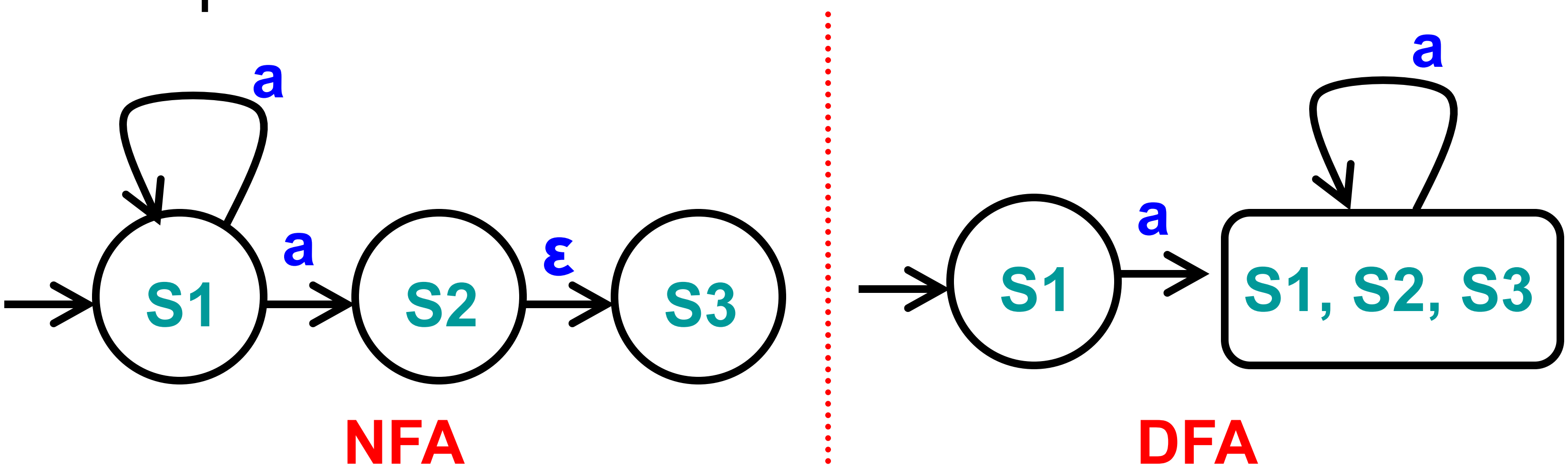
- How many states does <*A*> have?
  - Two added for each **|**, two added for each *****
  - O(n)
  - That's pretty good!

Reducing NFA to DFA



Reducing NFA to DFA

- NFA may be reduced to DFA
  - By explicitly tracking the set of NFA states
- Intuition
  - Build DFA where
    - Each DFA state represents a set of NFA "current states"
- Example



Algorithm for Reducing NFA to DFA

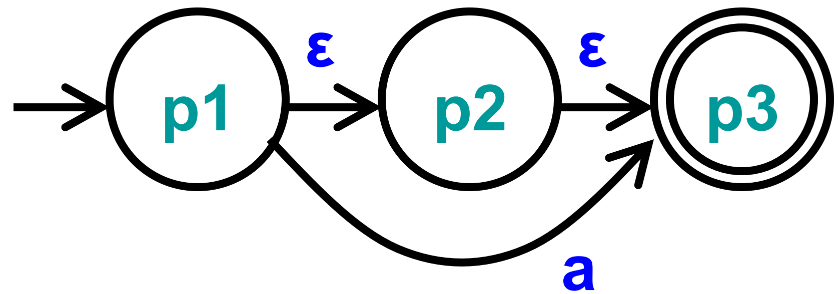
- Reduction applied using the subset algorithm
  - DFA state is a subset of set of all NFA states
- Algorithm
  - Input
    - NFA $(\Sigma, Q, q_0, F_n, \delta)$
  - Output
    - DFA $(\Sigma, R, r_0, F_d, \delta)$
  - Using two subroutines
    - ε-closure(δ, p) (and ε-closure(δ, Q))
    - move(δ, p, σ) (and move(δ, Q, σ))
      - (where p is an NFA state)

ε-transitions and ε-closure

- We say $p \xrightarrow{\varepsilon} q$
  - If it is possible to go from state p to state q by taking only ε-transitions in δ
  - If $\exists$ p, p_1, p_2, … p_n, q $\in$ Q such that
    - $\{p,\varepsilon,p_1\} \in \delta$, $\{p_1,\varepsilon,p_2\} \in \delta$, … , $\{p_n,\varepsilon,q\} \in \delta$
- ε-closure(δ, p)
  - Set of states reachable from p using ε-transitions alone
    - Set of states q such that $p \xrightarrow{\varepsilon} q$ according to δ
    - ε-closure(δ, p) = $\{q \mid p \xrightarrow{\varepsilon} q \text{ in } \delta\}$
    - ε-closure(δ, Q) = $\{ q \mid p \in Q,\ p \xrightarrow{\varepsilon} q \text{ in } \delta \}$
  - Notes
    - ε-closure(δ, p) always includes p
    - We write ε-closure(p) or ε-closure(Q) when δ is clear from context

ε-closure: Example 1

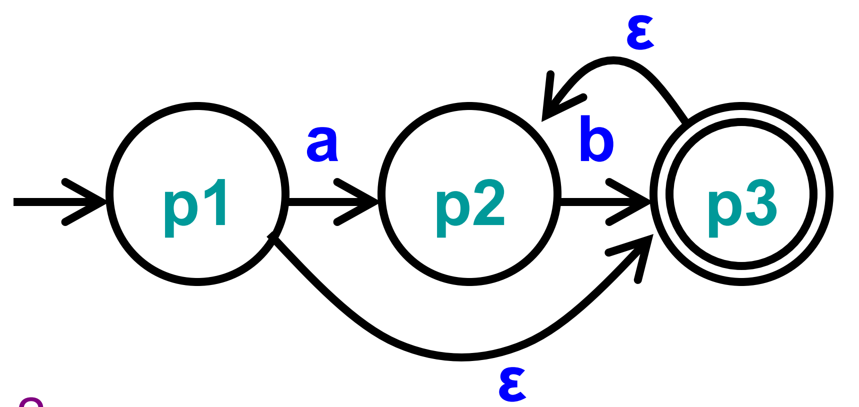
- Following NFA contains
  - $p1 \xrightarrow{\varepsilon} p2$
  - $p2 \xrightarrow{\varepsilon} p3$
  - $p1 \xrightarrow{\varepsilon} p3$
    - Since $p1 \xrightarrow{\varepsilon} p2$ and $p2 \xrightarrow{\varepsilon} p3$



- ε-closures
  - ε-closure(p1) = { p1, p2, p3 }
  - ε-closure(p2) = { p2, p3 }
  - ε-closure(p3) = { p3 }
  - ε-closure({ p1, p2 }) = { p1, p2, p3 } ∪ { p2, p3 }

ε-closure: Example 2

- Following NFA contains
  - $p1 \xrightarrow{\varepsilon} p3$
  - $p3 \xrightarrow{\varepsilon} p2$
  - $p1 \xrightarrow{\varepsilon} p2$
    - Since $p1 \xrightarrow{\varepsilon} p3$ and $p3 \xrightarrow{\varepsilon} p2$



- ε-closures
  - ε-closure(p1) = { p1, p2, p3 }
  - ε-closure(p2) = { p2 }
  - ε-closure(p3) = { p2, p3 }
  - ε-closure({ p2,p3 }) = { p2 } ∪ { p2, p3 }

ε-closure Algorithm: Approach

- Input: NFA $(\Sigma, Q, q_0, F_n, \delta)$, State Set R
- Output: State Set R'
- Algorithm

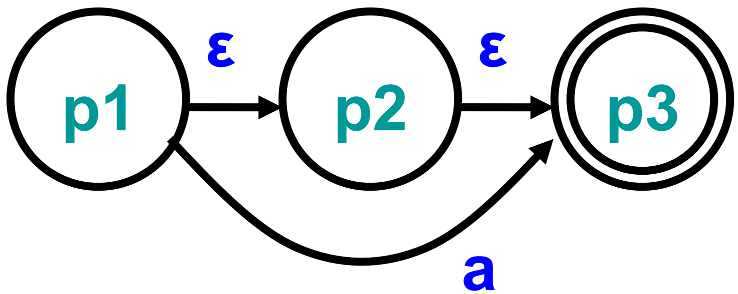
Let $R' = R$	// start states
Repeat	
Let $R = R'$	// continue from previous
Let $R' = R \cup \{q \mid p \in R, (p, \varepsilon, q) \in \delta\}$	// new ε-reachable states
Until $R = R'$	// stop when no new states

This algorithm computes a fixed point

ε-closure Algorithm Example

- Calculate ε-closure(δ,{p1})

R	R'
{p1}	{p1}
{p1}	{p1, p2}
{p1, p2}	{p1, p2, p3}
{p1, p2, p3}	{p1, p2, p3}



Let R' = R
Repeat
  Let R= R'
  Let R' = R $\cup$ {q | p $\in$ R, (p, ε, q) $\in$ δ}
Until R = R'

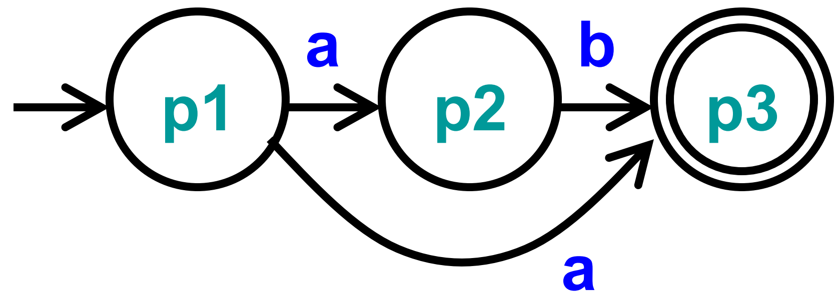
Calculating move(p,σ)

- move(δ,p,σ)
  - Set of states reachable from p using exactly one transition on symbol σ
    - Set of states q such that $\{p, \sigma, q\} \in \delta$
    - $\text{move}(\delta,p,\sigma) = \{ q \mid \{p, \sigma, q\} \in \delta \}$
    - $\text{move}(\delta,Q,\sigma) = \{ q \mid p \in Q, \{p, \sigma, q\} \in \delta \}$
      - i.e., can "lift" move() to a *set* of states Q

  - Notes:
    - move(δ,p,σ) is Ø if no transition $(p,\sigma,q) \in \delta$, for any q
    - We write move(p,σ) or move(R,σ) when δ clear from context

move(p,σ) : Example 1

- Following NFA
  - Σ = { a, b }

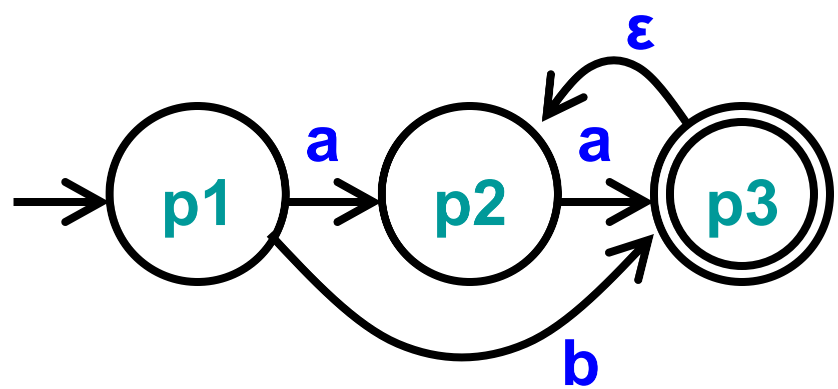


- Move
  - move(p1, a) = { p2, p3 }
  - move(p1, b) = Ø
  - move(p2, a) = Ø
  - move(p2, b) = { p3 }
  - move(p3, a) = Ø
  - move(p3, b) = Ø

move({p1,p2},b) = { p3 }

move(p,σ) : Example 2

- Following NFA
  - Σ = { a, b }



- Move
  - move(p1, a) = { p2 }
  - move(p1, b) = { p3 }
  - move(p2, a) = { p3 }
  - move(p2, b) = Ø
  - move(p3, a) = Ø
  - move(p3, b) = Ø

move({p1,p2},a) = {p2,p3}

NFA $\rightarrow$ DFA Reduction Algorithm ("subset")

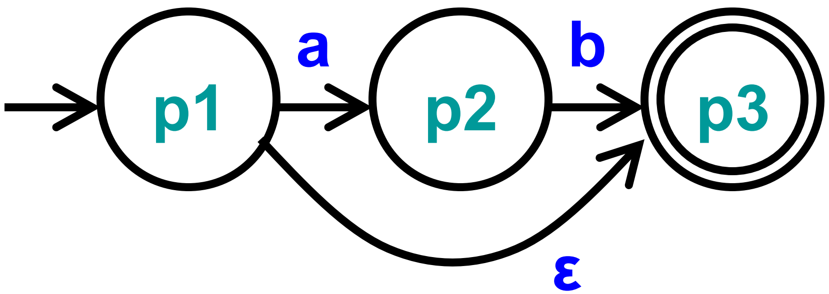
- Input NFA $(\Sigma, Q, q_0, F_n, \delta)$, Output DFA $(\Sigma, R, r_0, F_d, \delta')$
- Algorithm

Let r_0 = ε-closure(δ,q_0), add it to R	// DFA start state
While $\exists$ an unmarked state r $\in$ R	// process DFA state r
Mark r	// each state visited once
For each $\sigma \in \Sigma$	// for each symbol σ
Let E = move(δ,r,σ)	// states reached via σ
Let e = ε-closure(δ,E)	// states reached via ε
If e $\notin$ R	// if state e is new
Let R = R $\cup$ {e}	// add e to R (unmarked)
Let δ' = $\delta' \cup$ {r, σ, e}	// add transition r$\rightarrow$e on σ
Let F_d = {r | $\exists$ s $\in$ r with s $\in F_n$}	// final if include state in F_n

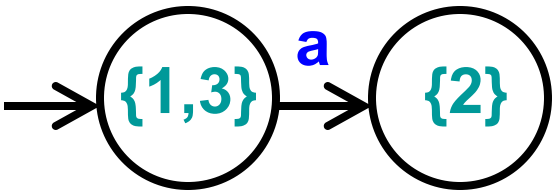
NFA $\rightarrow$ DFA Example 1

- Start = ε-closure(δ,p1) = { {p1,p3} }
- R = { {p1,p3} }
- r ∈ R = {p1,p3}
- move(δ,{p1,p3},a) = {p2}
  - e = ε-closure(δ,{p2}) = {p2}
  - R = R ∪ {{p2}} = { {p1,p3}, {p2} }
  - δ' = δ' ∪ {{p1,p3}, a, {p2}}
- move(δ,{p1,p3},b) = Ø

NFA

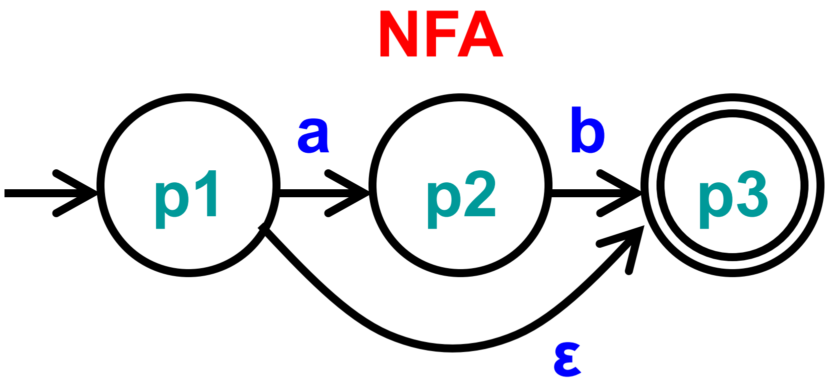


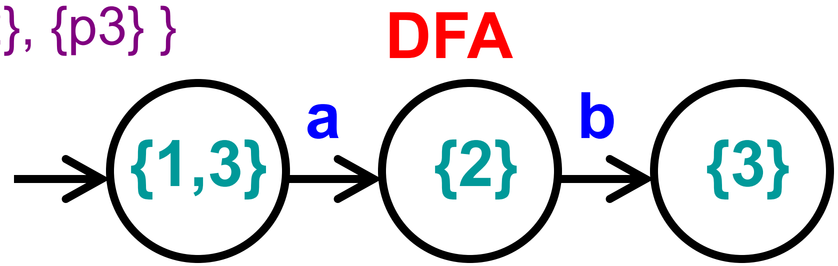
DFA



NFA → DFA Example 1 (cont.)

- R = { {p1,p3}, {p2} }
- r ∈ R = {p2}
- move(δ,{p2},a) = Ø
- move(δ,{p2},b) = {p3}
  - e = ε-closure(δ,{p3}) = {p3}
  - R = R ∪ {{p3}} = { {p1,p3}, {p2}, {p3} }
  - δ' = δ' ∪ {{p2}, b, {p3}}

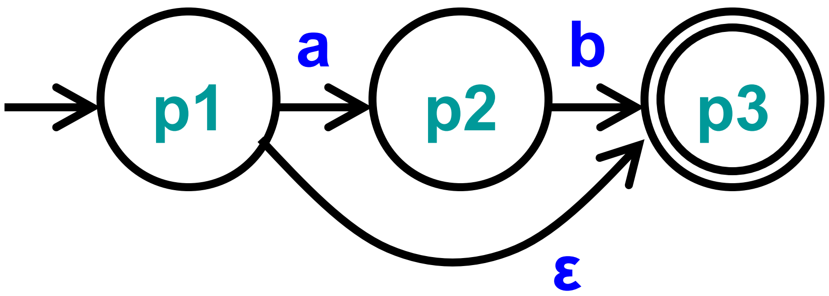




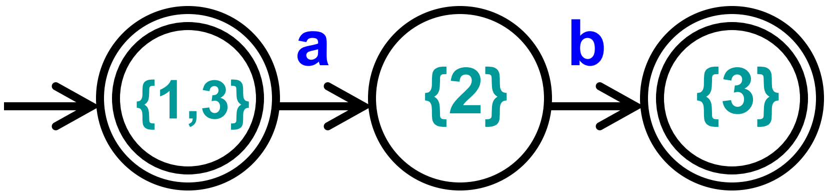
NFA → DFA Example 1 (cont.)

- R = { {p1,p3}, {p2}, {p3} }
- r ∈ R = {p3}
- Move({p3},a) = Ø
- Move({p3},b) = Ø
- Mark {p3}, exit loop
- F_d = {{p1,p3}, {p3}}
  - Since p3 ∈ F_n
- Done!

NFA

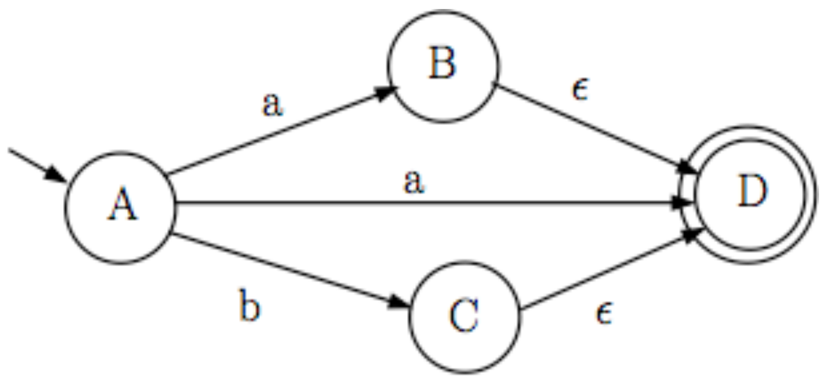


DFA

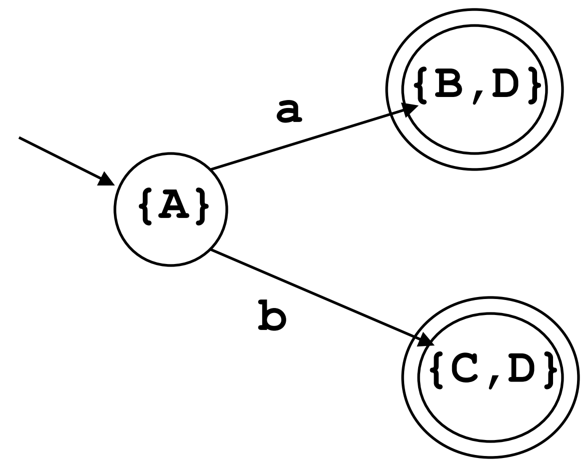


NFA → DFA Example 2

- NFA

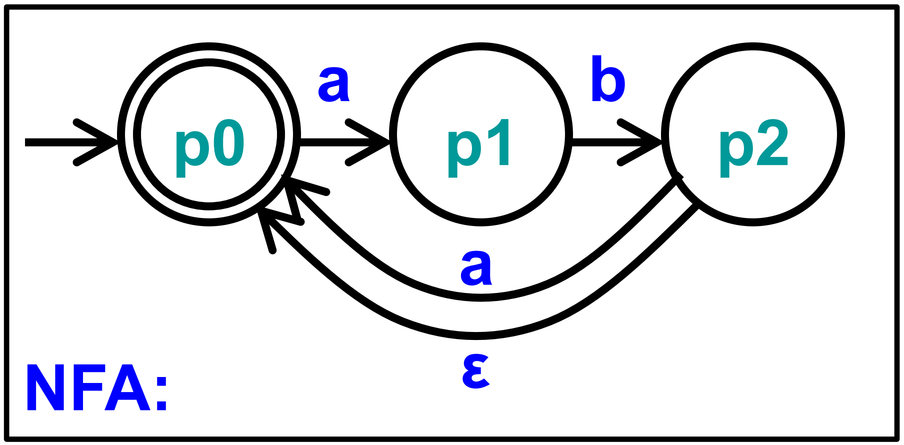


- DFA

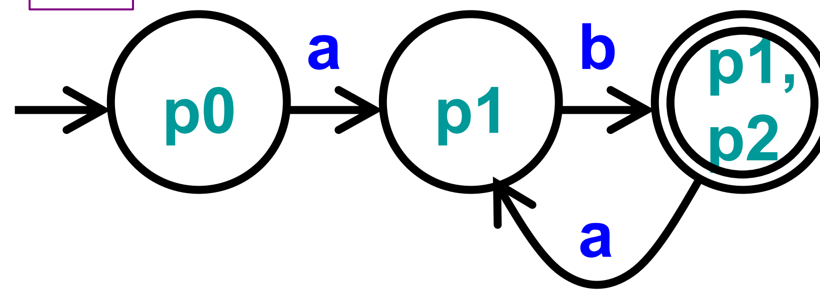


R = { {A}, {B,D}, {C,D} }

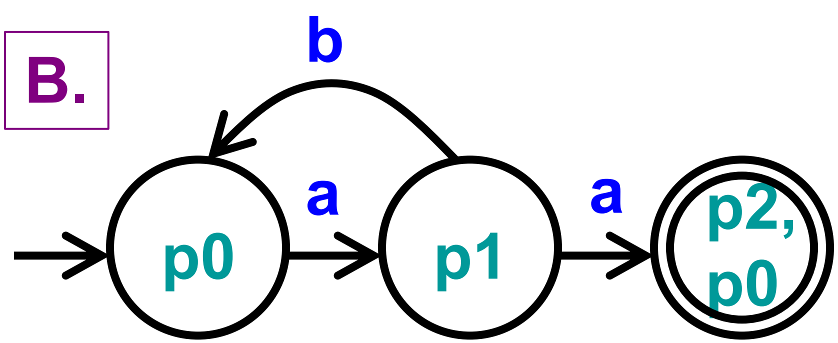
Quiz 4: Which DFA is equiv to this NFA?



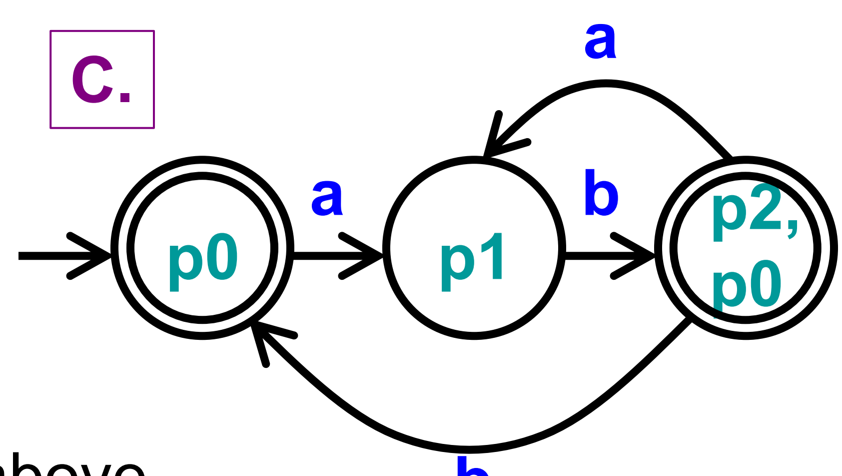
A.



B.

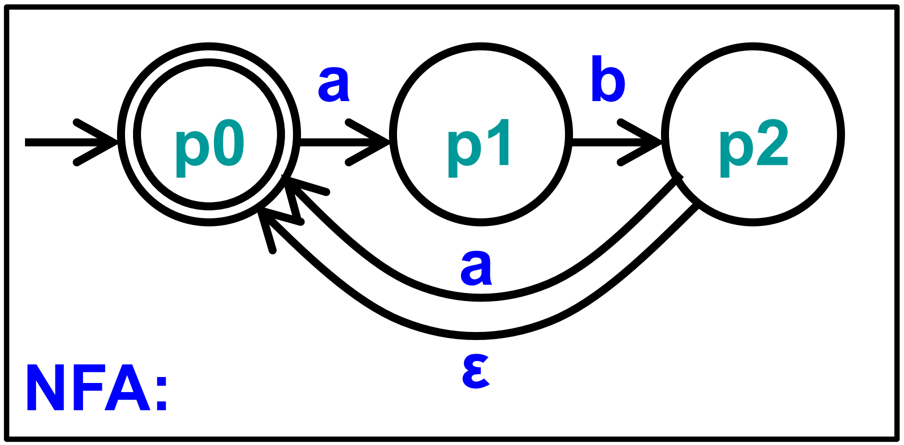


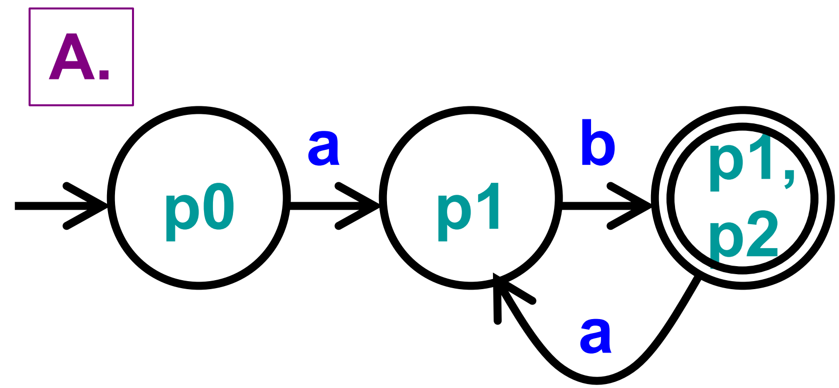
C.

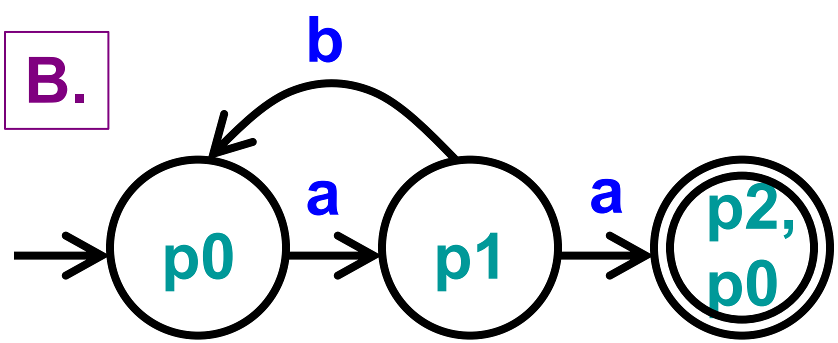


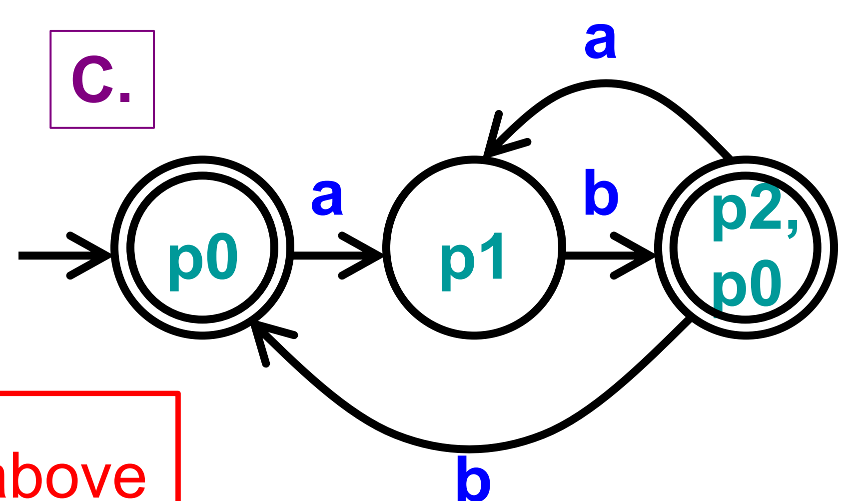
D. None of the above

Quiz 4: Which DFA is equiv to this NFA?



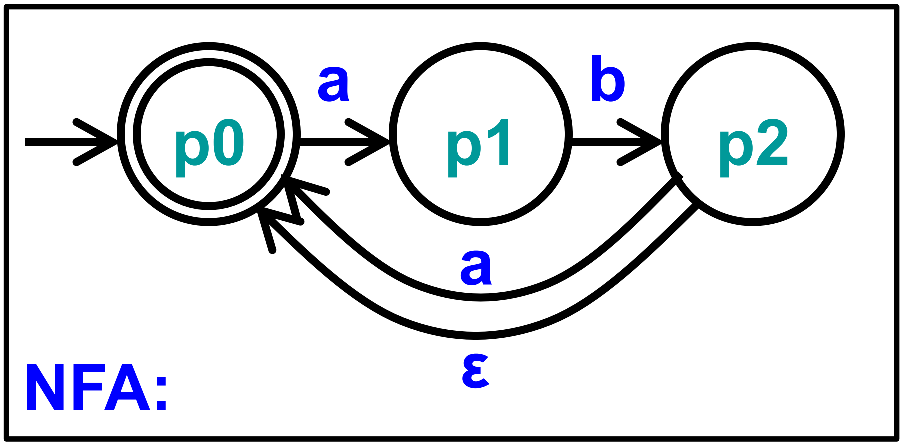


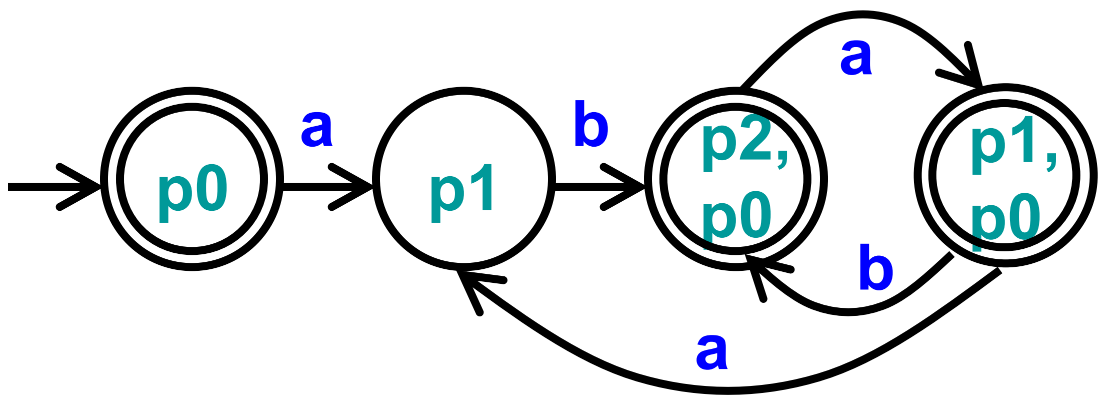




D. None of the above

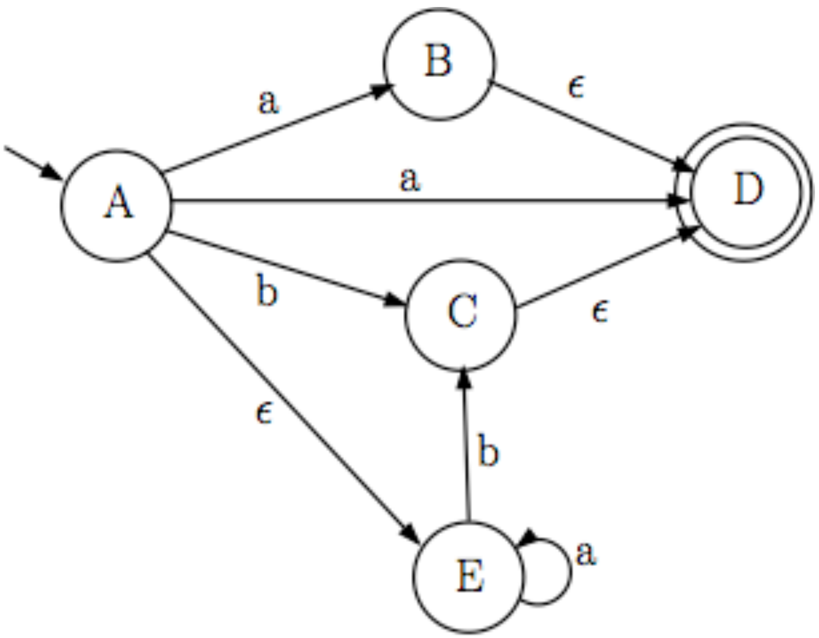
Actual Answer



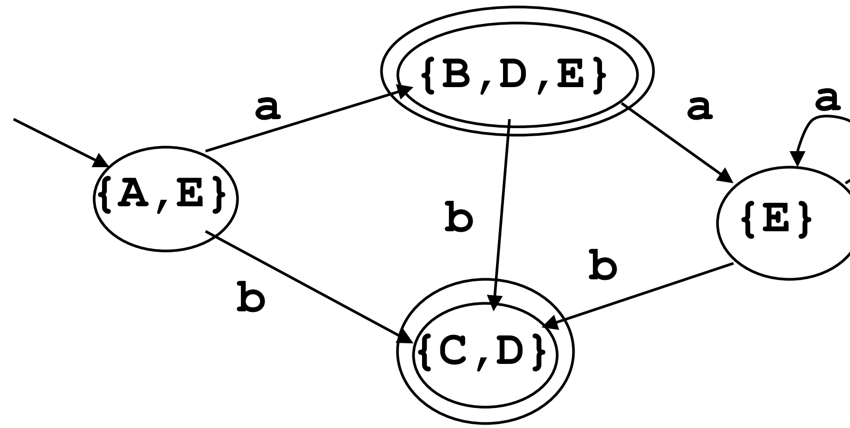


NFA → DFA Example 3

- NFA

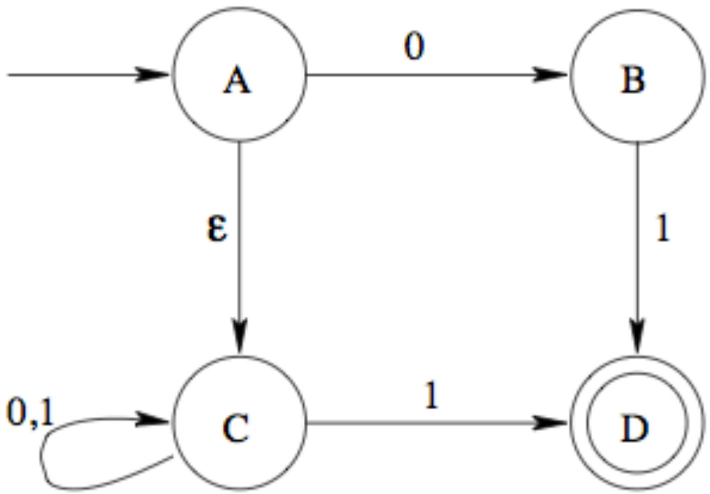


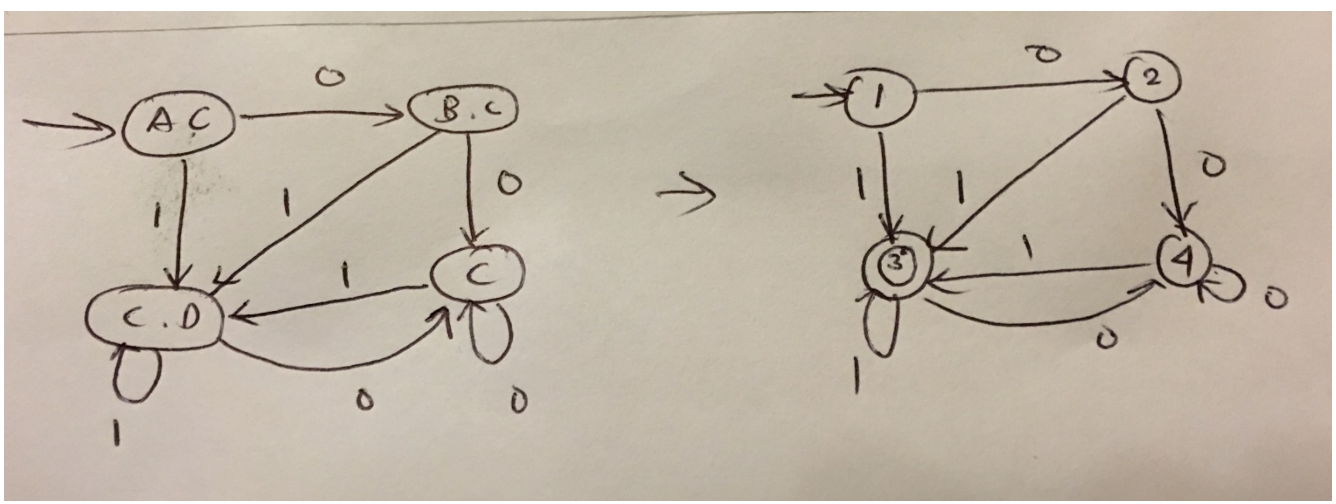
- DFA



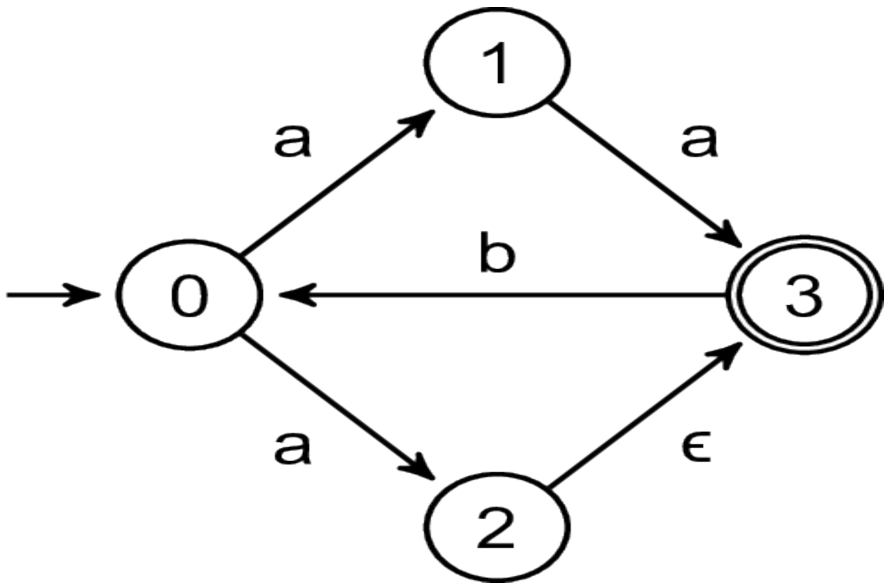
R = { {A,E}, {B,D,E}, {C,D}, {E} }

NFA → DFA Example

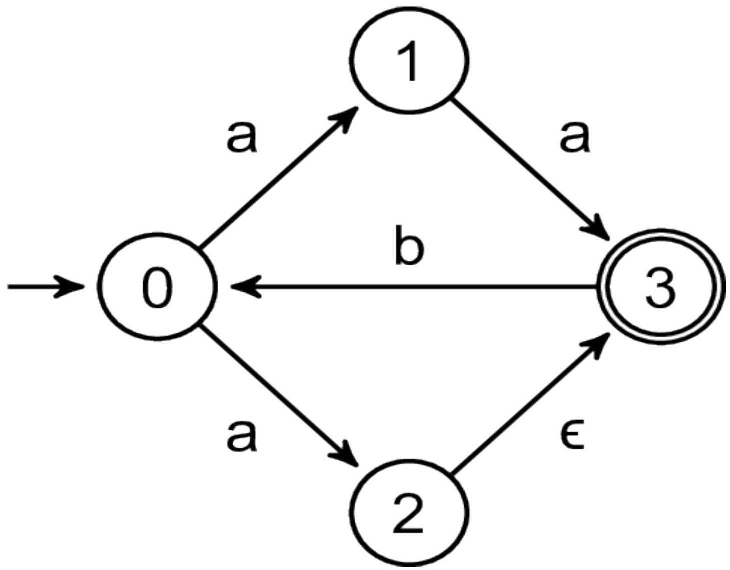


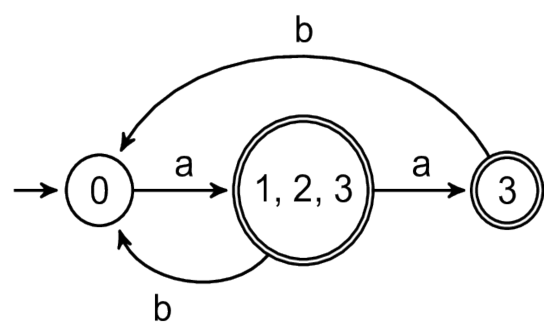


NFA → DFA Practice



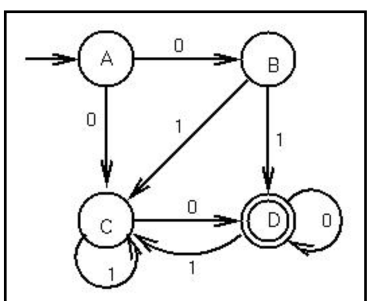
NFA → DFA Practice

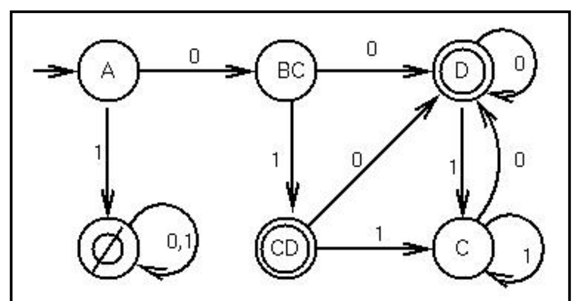




Analyzing the Reduction

- Can reduce any NFA to a DFA using subset alg.
- How many states in the DFA?
  - Each DFA state is a subset of the set of NFA states
  - Given NFA with n states, DFA may have 2^n states
    - Since a set with n items may have 2^n subsets
  - Corollary
    - Reducing a NFA with n states may be $O(2^n)$

NFA


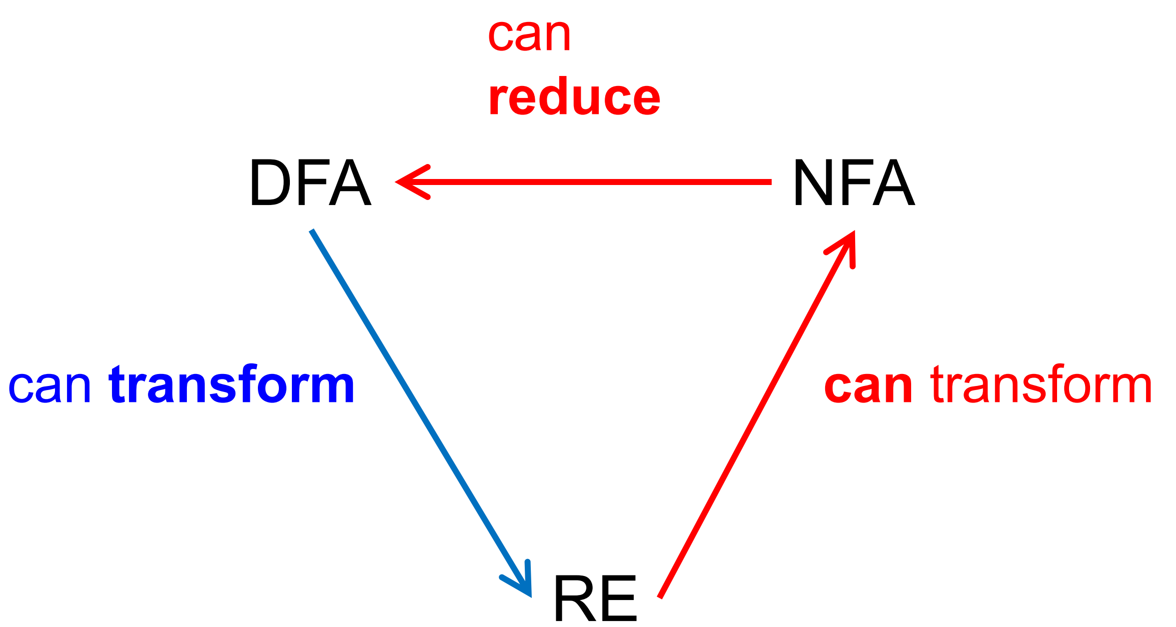


DFA

Recap: Matching a Regexp *R*

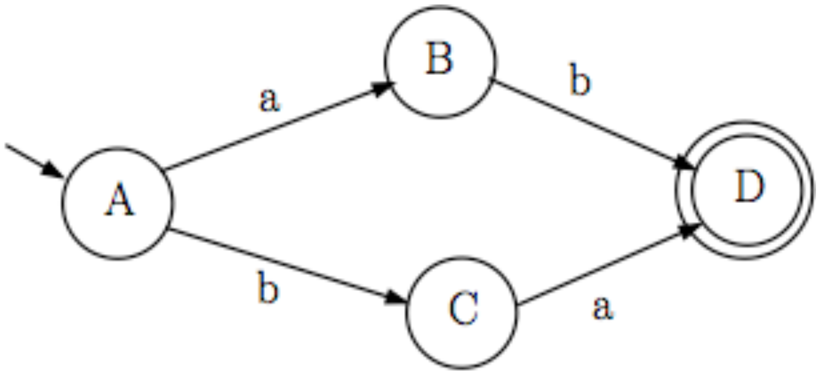
- Given *R*, construct NFA. Takes time $O(R)$
- Convert NFA to DFA. Takes time $O(2^{|R|})$
  - But usually not the worst case in practice
- Use DFA to accept/reject string s
  - Assume we can compute $\delta(q,\sigma)$ in constant time
  - Then time to process s is $O(|s|)$
    - Can't get much faster!
- Constructing the DFA is a one-time cost
  - But then processing strings is fast

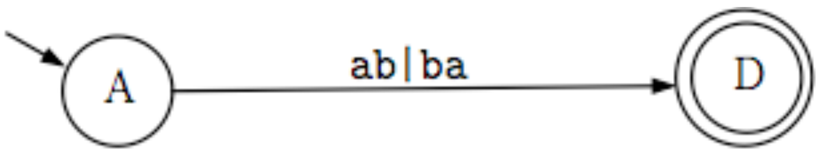
Closing the Loop: Reducing DFA to RE



Reducing DFAs to REs

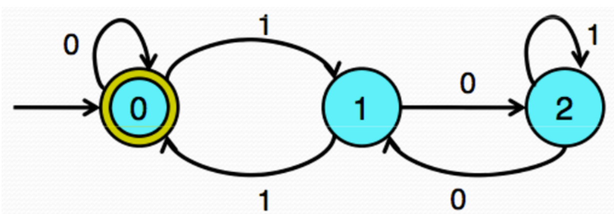
- General idea
  - Remove states one by one, labeling transitions with regular expressions
  - When two states are left (start and final), the transition label is the regular expression for the DFA

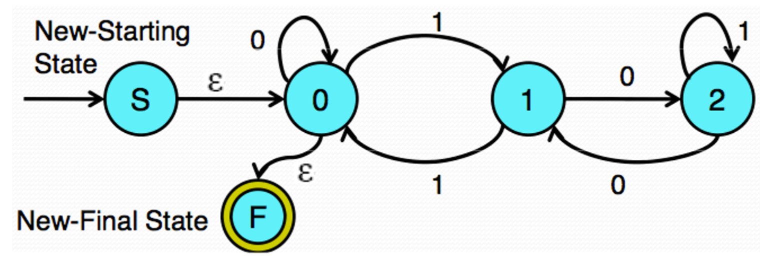


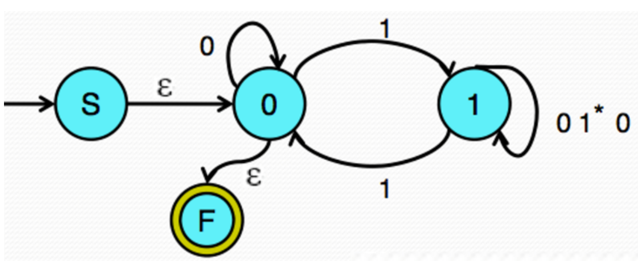


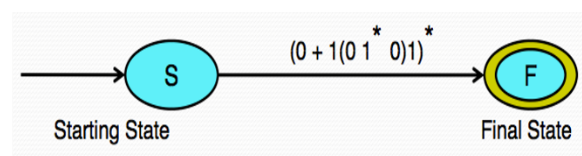
DFA to RE example

Language over $\Sigma = \{0,1\}$ such that every string is a multiple of 3 in binary



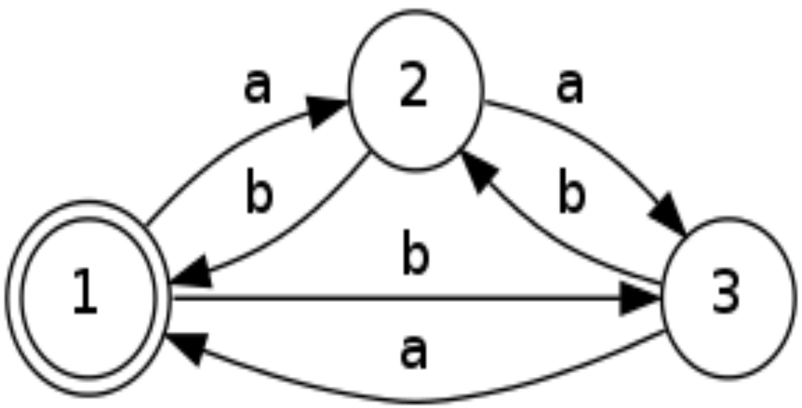




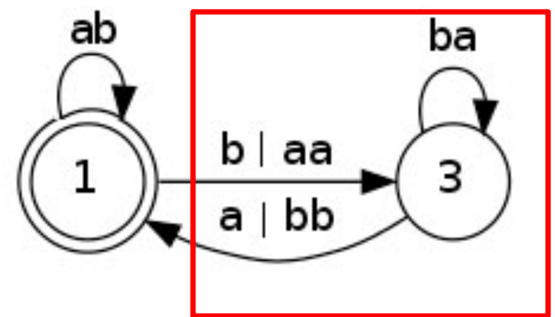


$$(0 + 1(0\,1^*\,0)1)^*$$

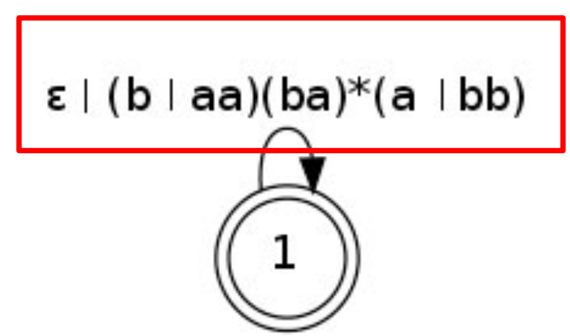
DFA to RE example



Remove 2



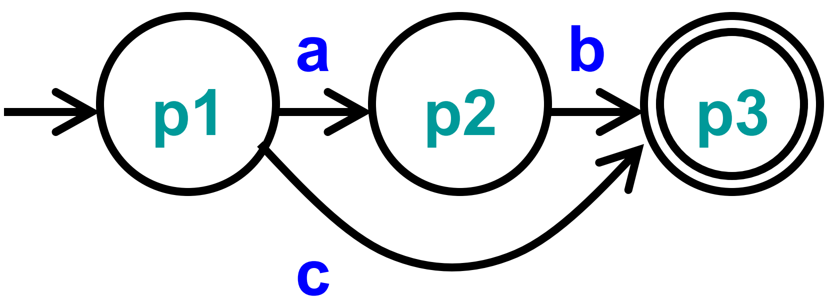
Remove 3

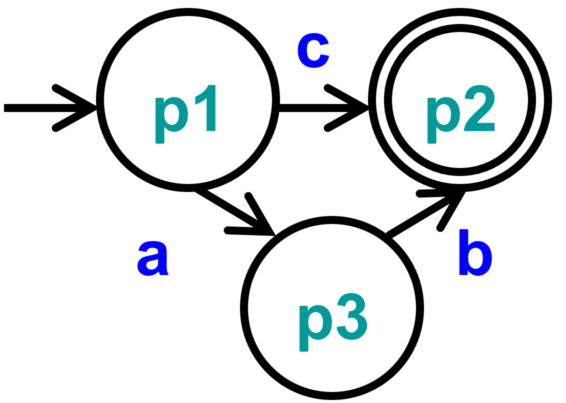


RE:$(ab \mid (b \mid aa)(ba)^{*}(a \mid bb))^{*}$

Minimizing DFAs

- Every regular language is recognizable by a unique minimum-state DFA
  - Ignoring the particular names of states
- In other words
  - For every DFA, there is a unique DFA with minimum number of states that accepts the same language





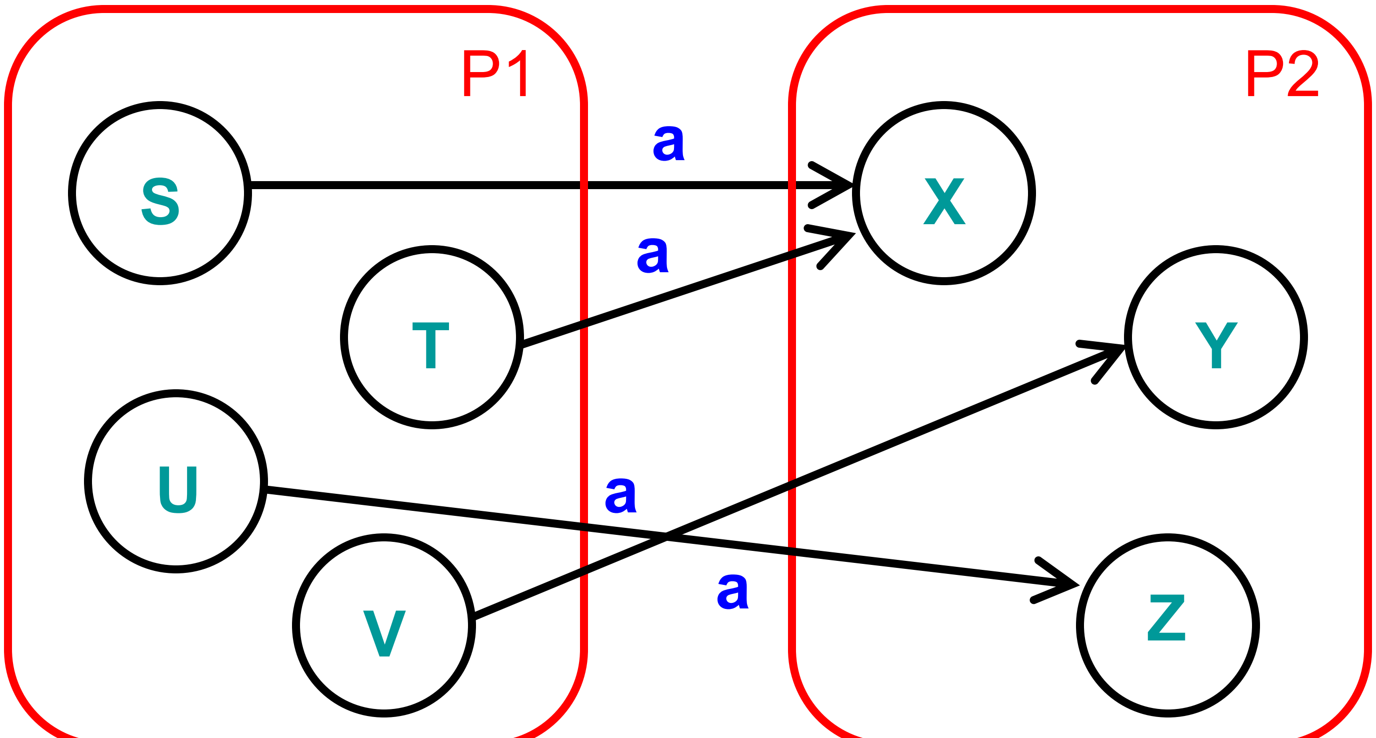
J. Hopcroft, "An n log n algorithm for minimizing states in a finite automaton," 1971

Minimizing DFA: Hopcroft Reduction

- Intuition
  - Look to distinguish states from each other
    - End up in different accept / non-accept state with identical input
- Algorithm
  - Construct initial partition
    - Accepting & non-accepting states
  - Iteratively split partitions (until partitions remain fixed)
    - Split a partition if **members in partition have transitions to different partitions for same input**
      - Two states x, y belong in same partition if and only if for all symbols in Σ they transition to the same partition
  - Update transitions & remove dead states

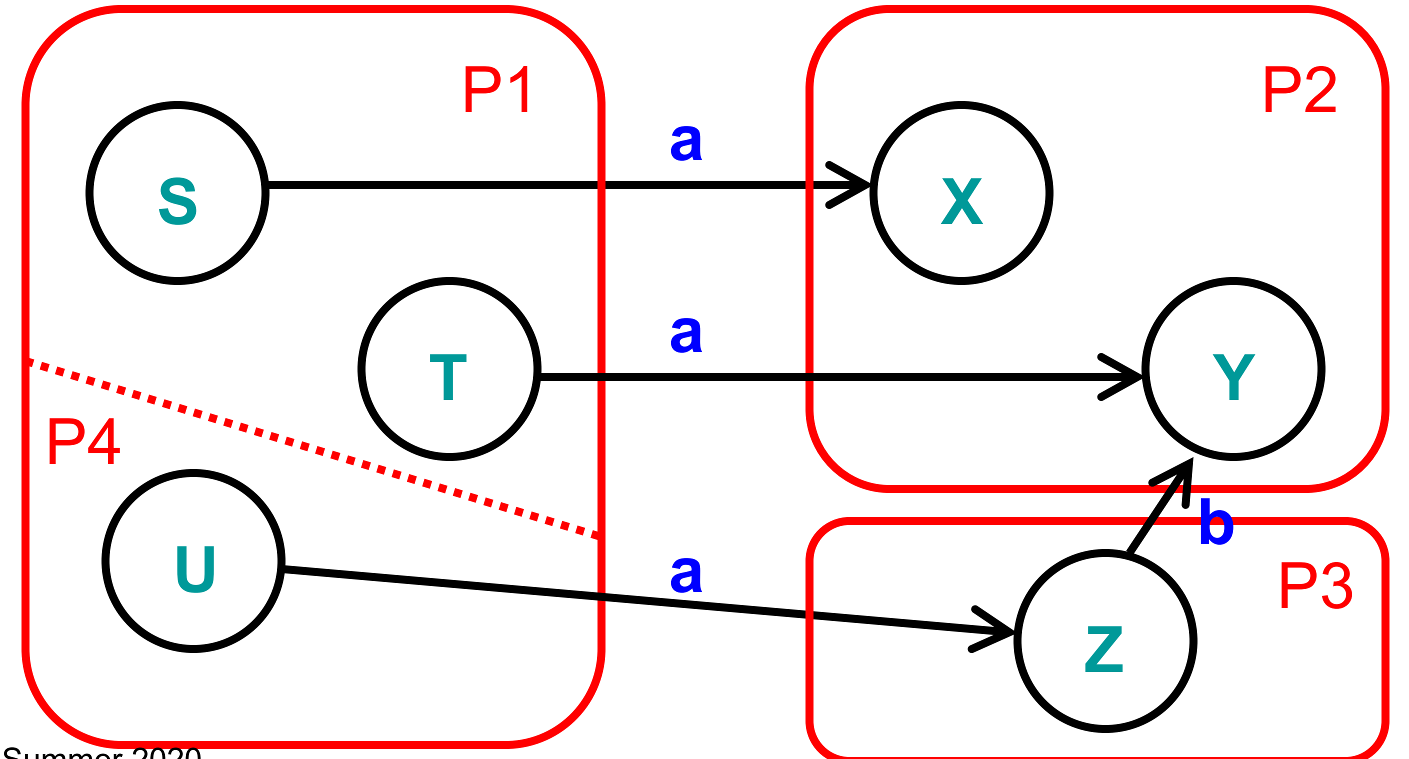
Splitting Partitions

- No need to split partition {S,T,U,V}
  - All transitions on a lead to identical partition P2
  - Even though transitions on a lead to different states



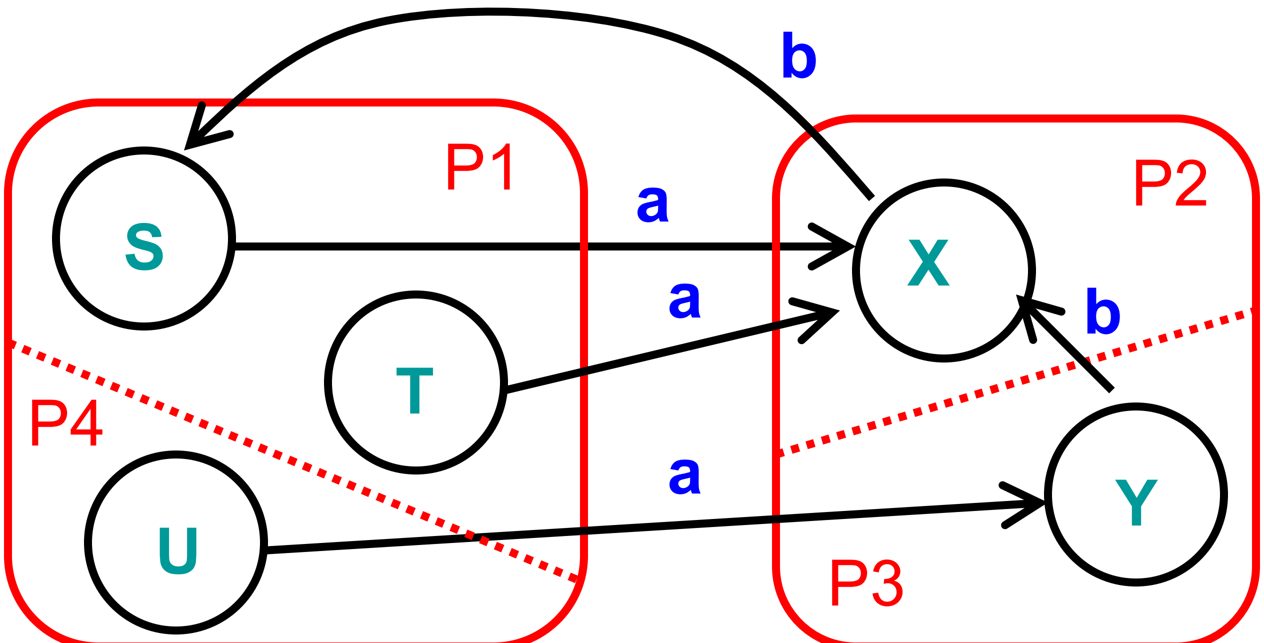
Splitting Partitions (cont.)

- Need to split partition {S,T,U} into {S,T}, {U}
  - Transitions on a from S,T lead to partition P2
  - Transition on a from U lead to partition P3



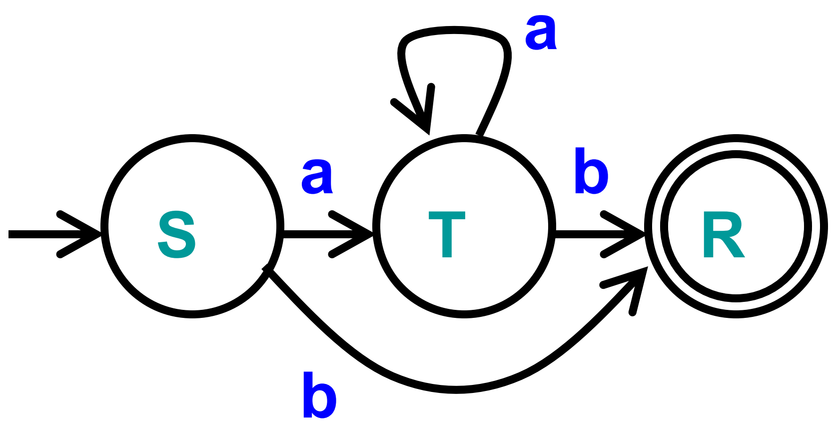
Resplitting Partitions

- Need to reexamine partitions after splits
  - Initially no need to split partition {S,T,U}
  - After splitting partition {X,Y} into {X}, {Y} we need to split partition {S,T,U} into {S,T}, {U}



Minimizing DFA: Example 1

- DFA

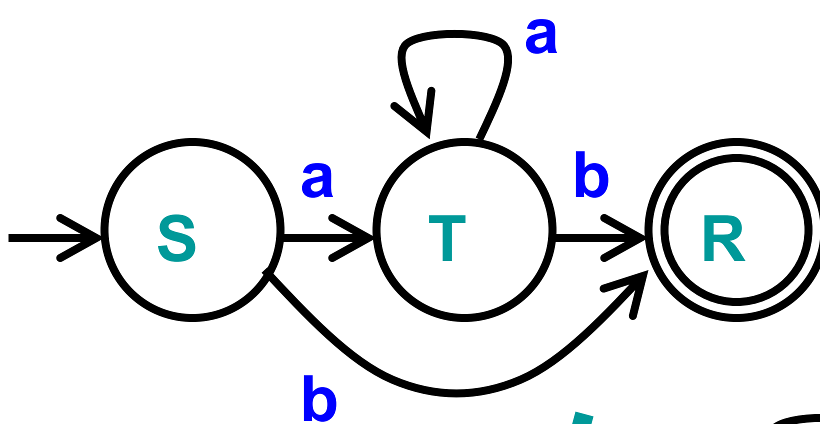


- Initial partitions

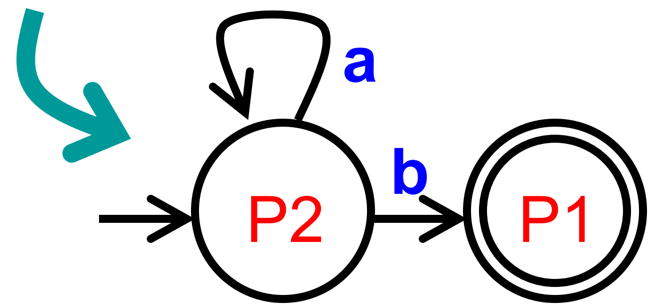
- Split partition

Minimizing DFA: Example 1

- DFA

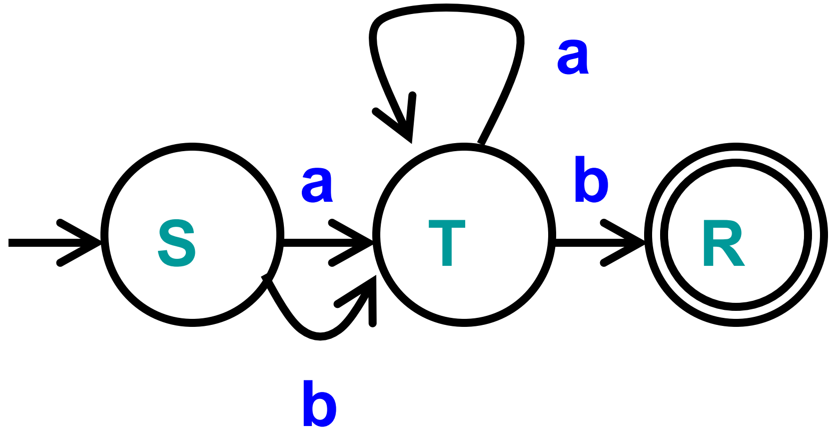


- Initial partitions
  - Accept { R } = P1
  - Reject { S, T } = P2



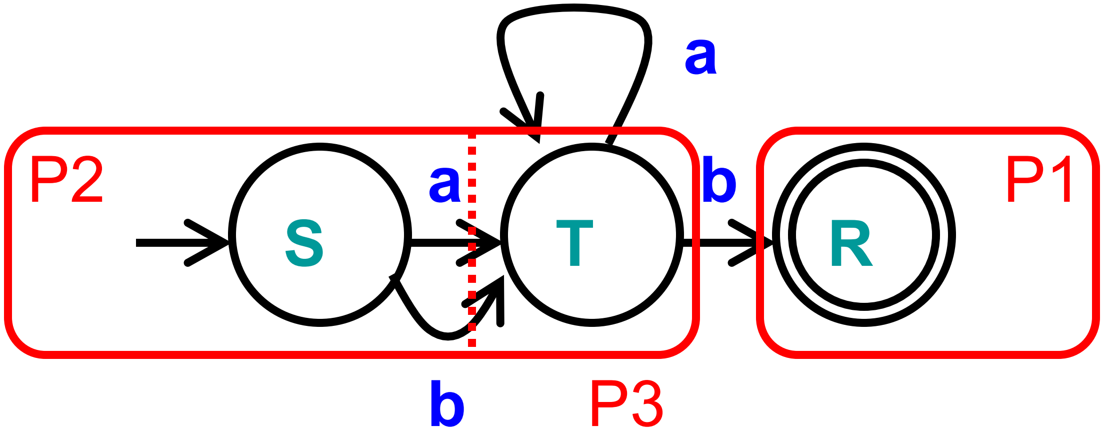
- Split partition? → Not required, minimization done
  - move(S,a) = T ∈ P2 – move(S,b) = R ∈ P1
  - move(T,a) = T ∈ P2 – move (T,b) = R ∈ P1

Minimizing DFA: Example 2



Minimizing DFA: Example 2

- DFA

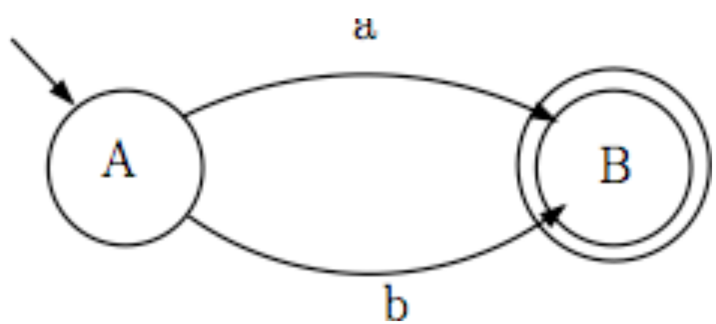


DFA already minimal

- Initial partitions
  - Accept { R } = P1
  - Reject { S, T } = P2
- Split partition? → Yes, different partitions for B
  - move(S,a) = T ∊ P2
    - move(S,b) = T ∊ P2
  - move(T,a) = T ∊ P2
    - move (T,b) = R ∊ P1

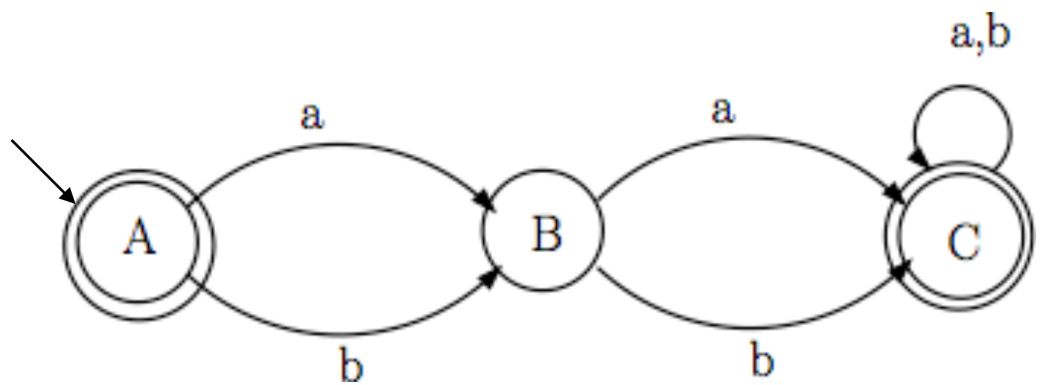
Complement of DFA

- Given a DFA accepting language L
  - How can we create a DFA accepting its complement?
  - Example DFA
    - $\Sigma = \{a,b\}$



Complement of DFA

- Algorithm
  - Add explicit transitions to a dead state
  - Change every accepting state to a non-accepting state & every non-accepting state to an accepting state
- Note this only works with DFAs
  - Why not with NFAs?



Summary of Regular Expression Theory

- Finite automata
  - DFA, NFA
- Equivalence of RE, NFA, DFA
  - RE $\rightarrow$ NFA
    - Concatenation, union, closure
  - NFA $\rightarrow$ DFA
    - ε-closure & subset algorithm
- DFA
  - Minimization, complementation