CMSC 330: Organization of Programming Languages

Operational Semantics

Formal Semantics of a Prog. Lang.

- Mathematical description of the meaning of programs written in that language
  - What a program computes, and what it does

- Three main approaches to formal semantics
  - Denotational
  - Operational
  - Axiomatic

This Course: Operational Semantics

- We will show how an operational semantics may be defined for Micro-Ocaml
  - And develop an interpreter for it, along the way

- Approach: use rules to define a judgment

$$e \Rightarrow v$$

  - Says "e *evaluates to* v"
  - e: expression in Micro-OCaml
  - v: value that results from evaluating e

Definitional Interpreter

- It turns out that the rules for judgment $e \Rightarrow v$ can be easily turned into idiomatic OCaml code
  - The language's expressions e and values v have corresponding OCaml datatype representations **`exp`** and **`value`**
  - The semantics is represented as a function

```
eval: exp -> value
```

- This way of presenting the semantics is referred to as a definitional interpreter
  - The interpreter defines the language's meaning

Micro-OCaml Expression Grammar

$$e ::= x \mid n \mid e + e \mid \texttt{let}\ x\ \texttt{=}\ e\ \texttt{in}\ e$$

- *e*, *x*, *n* are *meta-variables* that stand for categories of syntax
  - *x* is any identifier (like **`z`**, **`y`**, **`foo`**)
  - *n* is any numeral (like **`1`**, **`0`**, **`10`**, **`-25`**)
  - *e* is any expression (here defined, recursively!)

- *Concrete syntax* of actual expressions in **black**
  - Such as **`let`**, **`+`**, **`z`**, **`foo`**, **`in`**, …

- ::= and | are *meta-syntax* used to define the syntax of a language (part of "Backus-Naur form," or BNF)

Micro-OCaml Expression Grammar

$$e ::= x \mid n \mid e + e \mid \texttt{let } x \texttt{ = } e \texttt{ in } e$$

Examples

- `1` is a numeral *n* which is an expression *e*
- `1+z` is an expression *e* because
  - `1` is an expression *e*,
  - `z` is an identifier *x*, which is an expression *e*, and
  - *e* + *e* is an expression *e*
- `let z = 1 in 1+z` is an expression *e* because
  - `z` is an identifier *x*,
  - `1` is an expression *e* ,
  - `1+z` is an expression *e*, and
  - `let` *x* `=` *e* `in` *e* is an expression *e*

Abstract Syntax = Structure

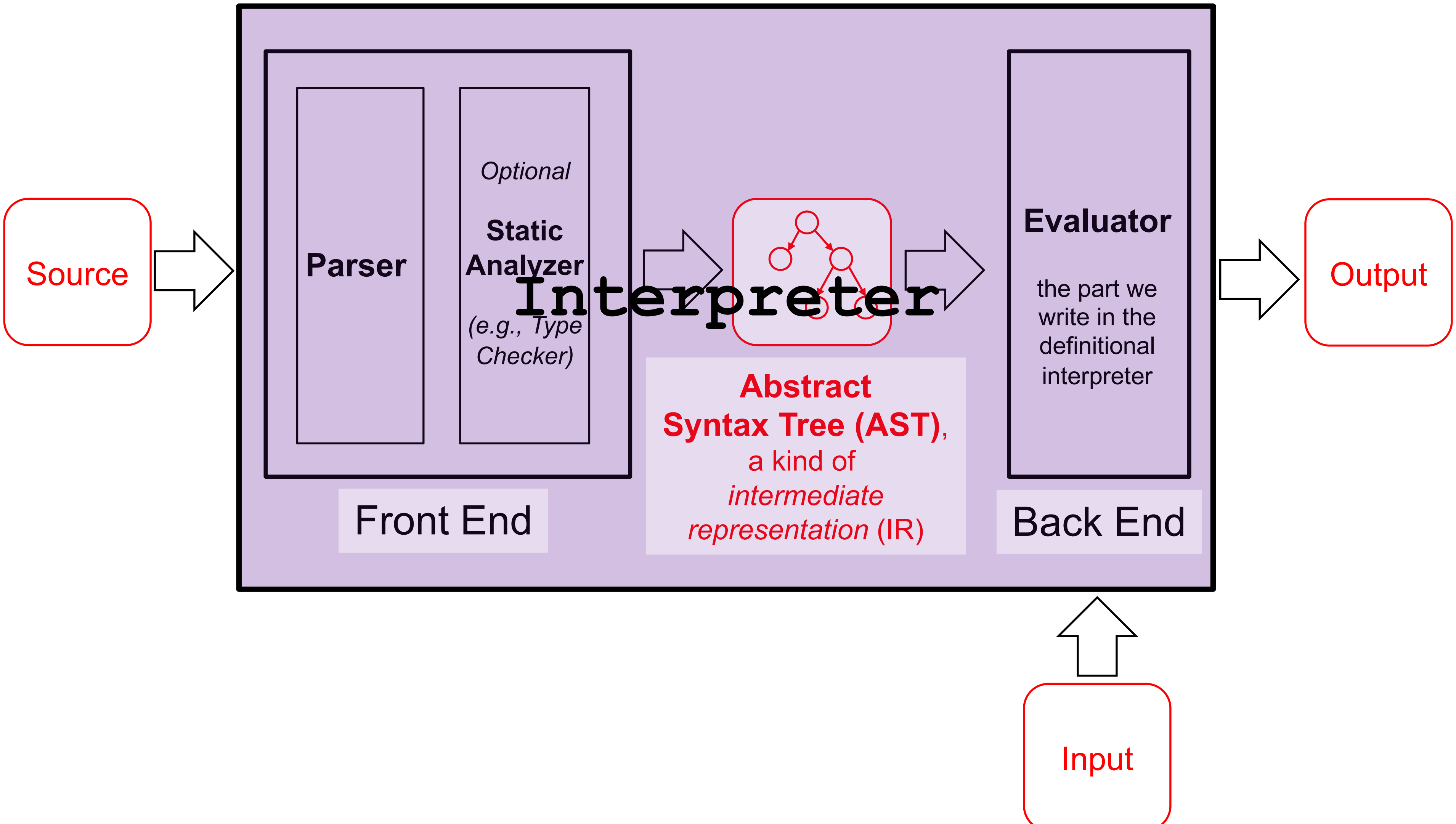
- Here, the grammar for ***e*** is describing its abstract syntax tree (AST), i.e., ***e***'s structure

$$e ::= x \mid n \mid e + e \mid \texttt{let } x \texttt{ = } e \texttt{ in } e$$

corresponds to (in definitional interpreter)

```
type id = string
type num = int
type exp =
    | Ident of id                 (* x *)
    | Num of num                  (* n *)
    | Plus of exp * exp         (* e+e *)
    | Let of id * exp * exp
                        (* let x=e in e *)
```

Aside: Real Interpreters



Values

- An expression's final result is a value. What can values be?

$$v ::= n$$

- Just numerals for now
  - In terms of an interpreter's representation:

    ```
    type value = int
    ```

  - In a full language, values v will also include booleans (`true`, `false`), strings, functions, …

Defining the Semantics

- Use rules to define judgment $e \Rightarrow v$
- Judgments are just statements. We use rules to prove that the statement is true.
  - `1+3` ⇒ `4`
    - `1+3` is an expression e, and `4` is a value v
    - This judgment claims that `1+3` evaluates to `4`
    - We use rules to prove it to be true
  - `let foo=1+2 in foo+5` ⇒ `8`
  - `let f=1+2 in let z=1 in f+z` ⇒ `4`

Rules as English Text

No rule when ***e*** is ***x***

- Suppose ***e*** is a numeral ***n***
  - Then ***e*** evaluates to itself, i.e., ***n*** ⇒ ***n***
- Suppose ***e*** is an addition expression ***e1*** + ***e2***
  - If ***e1*** evaluates to ***n1***, i.e., ***e1*** ⇒ ***n1***
  - If ***e2*** evaluates to ***n2***, i.e., ***e2*** ⇒ ***n2***
  - Then ***e*** evaluates to ***n3***, where ***n3*** is the sum of ***n1*** and ***n2***
  - I.e., ***e1*** + ***e2*** ⇒ ***n3***
- Suppose ***e*** is a let expression `let` ***x*** `=` ***e1*** `in` ***e2***
  - If ***e1*** evaluates to ***v***, i.e., ***e1*** ⇒ ***v1***
  - If ***e2{v1/x}*** evaluates to ***v2***, i.e., ***e2{v1/x}*** ⇒ ***v2***
    - Here, ***e2{v1/x}*** means "the expression after substituting occurrences of ***x*** in ***e2*** with ***v1***"
  - Then ***e*** evaluates to ***v2***, i.e., `let` ***x*** `=` ***e1*** `in` ***e2*** ⇒ ***v2***

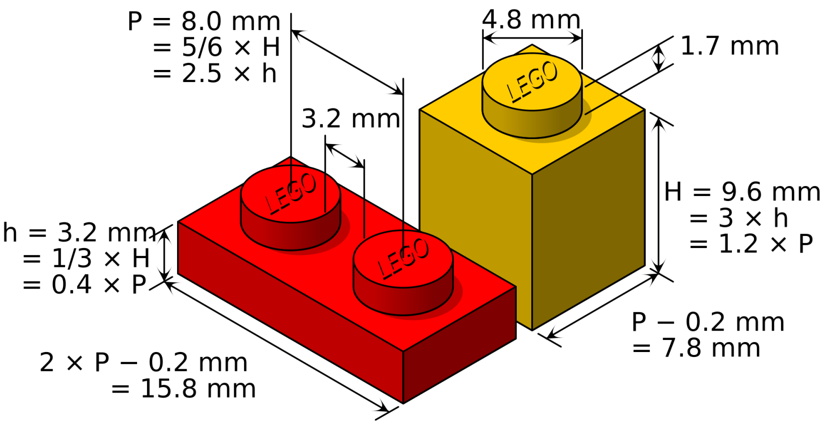
Rules of Inference

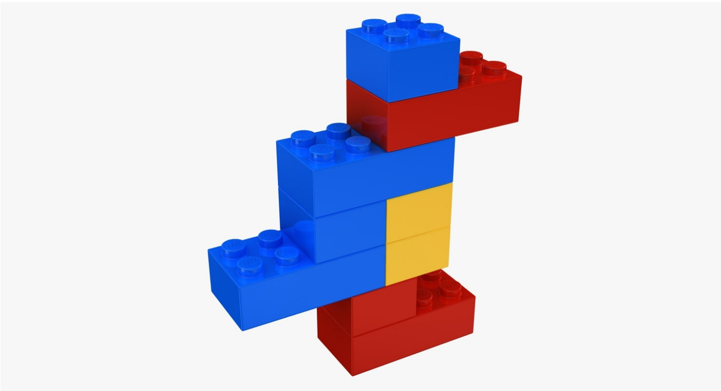
- We can use a more compact notation for the rules we just presented: rules of inference
  - Has the following format

$$\frac{H_1 \quad \ldots \quad H_n}{C}$$

  - Says: if the conditions $H_1 \quad \ldots \quad H_n$ ("hypotheses") are true, then the condition C ("conclusion") is true
  - If n=0 (no hypotheses) then the conclusion automatically holds; this is called an axiom
- We are using inference rules where C is our judgment about evaluation, i.e., that $e \Rightarrow v$

Lego Blocks and Lego Cars





Rules of Inference: Num and Sum

- Suppose **e** is a numeral **n**
  - Then **e** evaluates to itself, i.e., **n** ⇒ **n**

$$\frac{}{n \Rightarrow n}$$

- Suppose **e** is an addition expression **e1** + **e2**
  - If **e1** evaluates to **n1**, i.e., **e1** ⇒ **n1**
  - If **e2** evaluates to **n2**, i.e., **e2** ⇒ **n2**
  - Then **e** evaluates to **n3**, where **n3** is the sum of **n1** and **n2**
  - I.e., **e1** + **e2** ⇒ **n3**

$$\frac{e1 \Rightarrow n1 \qquad e2 \Rightarrow n2 \qquad n3 \text{ is } n1+n2}{e1 + e2 \Rightarrow n3}$$

Rules of Inference: Let

- Suppose ***e*** is a let expression `let` ***x*** `=` ***e1*** `in` ***e2***
  - If ***e1*** evaluates to ***v***, i.e., ***e1*** ⇒ ***v1***
  - If ***e2{v1/x}*** evaluates to ***v2***, i.e., ***e2{v1/x}*** ⇒ ***v2***
  - Then ***e*** evaluates to ***v2***, i.e., `let` ***x*** `=` ***e1*** `in` ***e2*** ⇒ ***v2***

$$\frac{e1 \Rightarrow v1 \qquad e2\{v1/x\} \Rightarrow v2}{\texttt{let } x = e1 \texttt{ in } e2 \Rightarrow v2}$$

Derivations

- When we apply rules to an expression in succession, we produce a derivation
  - It's a kind of tree, rooted at the conclusion
- Produce a derivation by goal-directed search
  - Pick a rule that could prove the goal
  - Then repeatedly apply rules on the corresponding hypotheses
    - Goal: Show that `let x = 4 in x+3` ⇒ `7`

Derivations

$$\frac{}{n \Rightarrow n}$$

$$\frac{e1 \Rightarrow n1 \quad e2 \Rightarrow n2 \quad n3 \text{ is } n1+n2}{e1 + e2 \Rightarrow n3}$$

$$\frac{e1 \Rightarrow v1 \quad e2\{v1/x\} \Rightarrow v2}{\texttt{let } x = e1 \texttt{ in } e2 \Rightarrow v2}$$

Goal: show that

`let x = 4 in x+3` ⇒ `7`

$$\frac{4 \Rightarrow 4 \qquad \dfrac{4 \Rightarrow 4 \quad 3 \Rightarrow 3 \quad 7 \text{ is } 4+3}{4+3 \Rightarrow 7}}{\texttt{let x = 4 in x+3} \Rightarrow 7}$$

Quiz 1

What is derivation of the following judgment?

```
2 + (3 + 8) ⇒ 13
```

```
(a)
2 ⇒ 2      3 + 8 ⇒ 11
---------------------
2 + (3 + 8) ⇒ 13
```

```
(b)
3 ⇒ 3    8 ⇒ 8
------------
3 + 8 ⇒ 11           2 ⇒ 2
-------------------------
2 + (3 + 8) ⇒ 13
```

```
(c)
          8 ⇒ 8
          3 ⇒ 3
          11 is 3+8
          ---------
2 ⇒ 2     3 + 8 ⇒ 11     13 is 2+11
----------------------------------
2 + (3 + 8) ⇒ 13
```

Quiz 1

What is derivation of the following judgment?

`2 + (3 + 8) ⇒ 13`

(a)

```
2 ⇒ 2       3 + 8 ⇒ 11
-------------------------
2 + (3 + 8) ⇒ 13
```

(b)

```
3 ⇒ 3     8 ⇒ 8
--------------
3 + 8 ⇒ 11              2 ⇒ 2
-----------------------------
2 + (3 + 8) ⇒ 13
```

(c)

```
            8 ⇒ 8
            3 ⇒ 3
            11 is 3+8
            ----------
2 ⇒ 2       3 + 8 ⇒ 11       13 is 2+11
----------------------------------------
2 + (3 + 8) ⇒ 13
```

Definitional Interpreter

Trace of evaluation of **`eval`** function corresponds to a derivation by the rules

- The style of rules lends itself directly to the implementation of an interpreter as a recursive function

```
let rec eval (e:exp):value =
  match e with
    Ident x -> (* no rule *)
     failwith "no value"
   | Num n -> n
   | Plus (e1,e2) ->
     let n1 = eval e1 in
     let n2 = eval e2 in
     let n3 = n1+n2 in
     n3
   | Let (x,e1,e2) ->
     let v1 = eval e1 in
     let e2' = subst v1 x e2 in
     let v2 = eval e2' in v2
```

$$\frac{}{n \Rightarrow n}$$

$$\frac{e1 \Rightarrow n1 \quad e2 \Rightarrow n2 \quad n3 \text{ is } n1+n2}{e1 + e2 \Rightarrow n3}$$

$$\frac{e1 \Rightarrow v1 \quad e2\{v1/x\} \Rightarrow v2}{\texttt{let } x = e1 \texttt{ in } e2 \Rightarrow v2}$$

Derivations = Interpreter Call Trees

$$\dfrac{\mathtt{4} \Rightarrow \mathtt{4} \qquad \dfrac{\mathtt{4} \Rightarrow \mathtt{4} \quad \mathtt{3} \Rightarrow \mathtt{3} \quad \mathtt{7}\text{ is }\mathtt{4+3}}{\mathtt{4+3} \Rightarrow \mathtt{7}}}{\mathtt{let\ x\ =\ 4\ in\ x+3} \Rightarrow \mathtt{7}}$$

Has the same shape as the recursive call tree of the interpreter:

$$\dfrac{\mathtt{eval\ Num\ 4} \Rightarrow 4 \qquad \dfrac{\mathtt{eval\ Num\ 4} \Rightarrow 4 \quad \mathtt{eval\ Num\ 3} \Rightarrow 3 \quad \mathtt{7}\text{ is }\mathtt{4+3}}{\begin{array}{l}\mathtt{eval\ (subst\ 4\ ``x"} \\ \quad \mathtt{Plus(Ident(``x"),Num\ 3))} \Rightarrow 7\end{array}}}{\mathtt{eval\ Let(``x",Num\ 4,Plus(Ident(``x"),Num\ 3))} \Rightarrow 7}$$

Semantics Defines Program Meaning

- $e \Rightarrow v$ holds if and only if a *proof* can be built
  - Proofs are derivations: axioms at the top, then rules whose hypotheses have been proved to the bottom
  - No proof means $e \not\Rightarrow v$
- Proofs can be constructed bottom-up
  - In a goal-directed fashion
- Thus, function eval $e = \{v \mid e \Rightarrow v\}$
  - Determinism of semantics implies at most one element for any e
- So: Expression e *means* v

Environment-style Semantics

- The previous semantics uses substitution to handle variables
  - As we evaluate, we replace all occurrences of a variable ***x*** with values it is bound to
- An alternative semantics, closer to a real implementation, is to use an environment
  - As we evaluate, we maintain an explicit map from variables to values, and look up variables as we see them

Environments

- Mathematically, an environment is a partial function from identifiers to values
  - If A is an environment, and x is an identifier, then A(x) can either be …
  - … a value (intuition: the variable has been declared)
  - … or undefined (intuition: variable has not been declared)
- An environment can also be thought of as a table
  - If A is

Id	**Val**
`x`	0
`y`	2

  - then A(`x`) is `0`, A(`y`) is `2`, and A(`z`) is undefined

Notation, Operations on Environments

- $\bullet$ is the empty environment (undefined for all ids)
- If A is an environment then A,$\mathtt{x}$:$\mathtt{v}$ is one that extends A with a mapping from $\mathtt{x}$ to $\mathtt{v}$
  - Sometimes just write $\mathtt{x}$:$\mathtt{v}$ instead of $\bullet$,$\mathtt{x}$:$\mathtt{v}$ for brevity
  - *NB.* if A maps $\mathtt{x}$ to some $\mathtt{v}'$, then that mapping is *shadowed* by the mapping $\mathtt{x}$:$\mathtt{v}$
- Lookup A($\mathtt{x}$) is defined as follows

$$\bullet(\mathtt{x}) = \text{undefined}$$

$$(A,\ \mathtt{y}{:}\mathtt{v})(\mathtt{x}) = \begin{cases} \mathtt{v} & \text{if } \mathtt{x} = \mathtt{y} \\ A(\mathtt{x}) & \text{if } \mathtt{x} <> \mathtt{y} \text{ and } A(\mathtt{x}) \text{ defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Definitional Interpreter: Environments

```
type env = (id * value) list

let extend env x v = (x,v)::env

let rec lookup env x =
  match env with
    [] -> failwith "undefined"
  | (y,v)::env' ->
    if x = y then v
    else lookup env' x
```

An environment is just a list of mappings, which are just pairs of variable to value
- called an association list

Semantics with Environments

- The environment semantics changes the judgment

$$e \Rightarrow v$$

to be

$$A;\ e \Rightarrow v$$

where A is an environment
  - Idea: A is used to give values to the identifiers in e
  - A can be thought of as containing declarations made up to e
- Previous rules can be modified by
  - Inserting A everywhere in the judgments
  - Adding a rule to look up variables x in A
  - Modifying the rule for `let` to add x to A

Environment-style Rules

$$\frac{A(x) = v}{A;\ x \Rightarrow v}$$

Look up variable x in environment A

$$\frac{}{A;\ n \Rightarrow n}$$

$$\frac{A;\ e1 \Rightarrow v1 \qquad A,x{:}v1;\ e2 \Rightarrow v2}{A;\ \texttt{let}\ x = e1\ \texttt{in}\ e2 \Rightarrow v2}$$

Extend environment A with mapping from x to $v1$

$$\frac{A;\ e1 \Rightarrow n1 \qquad A;\ e2 \Rightarrow n2 \qquad n3 \text{ is } n1{+}n2}{A;\ e1 + e2 \Rightarrow n3}$$

Definitional Interpreter: Evaluation

```
let rec eval env e =
  match e with
    Ident x -> lookup env x
  | Num n -> n
  | Plus (e1,e2) ->
     let n1 = eval env e1 in
     let n2 = eval env e2 in
     let n3 = n1+n2 in
     n3
  | Let (x,e1,e2) ->
     let v1 = eval env e1 in
     let env' = extend env x v1 in
     let v2 = eval env' e2 in v2
```

Quiz 2

What is a derivation of the following judgment?

$$\bullet;\ \texttt{let x=3 in x+2} \Rightarrow 5$$

(a)

```
          x ⇒ 3    2 ⇒ 2   5 is 3+2
          ---------------------
3 ⇒ 3      x+2 ⇒ 5
------------------------
let x=3 in x+2 ⇒ 5
```

(c)

```
x:2; x⇒3  x:2; 2⇒2   5 is 3+2
------------------------------
•; let x=3 in x+2 ⇒ 5
```

(b)

```
           x:3; x ⇒ 3  x:3; 2 ⇒ 2   5 is 3+2
           ---------------------------------
•;3 ⇒ 3     x:3; x+2 ⇒ 5
-------------------------
•; let x=3 in x+2 ⇒ 5
```

Quiz 2

What is a derivation of the following judgment?

```
•; let x=3 in x+2 ⇒ 5
```

```
(a)
          x ⇒ 3    2 ⇒ 2    5 is 3+2
          ---------------------
3 ⇒ 3      x+2 ⇒ 5
------------------------
let x=3 in x+2 ⇒ 5
```

```
(c)
x:2; x⇒3   x:2; 2⇒2   5 is 3+2
-------------------------------
•; let x=3 in x+2 ⇒ 5
```

```
(b)          x:3; x ⇒ 3   x:3; 2 ⇒ 2    5 is 3+2
             -----------------------------------
•;3 ⇒ 3      x:3; x+2 ⇒ 5
--------------------------
•; let x=3 in x+2 ⇒ 5
```

Adding Conditionals to Micro-OCaml

$$e ::= x \mid v \mid e + e \mid \texttt{let } x = e \texttt{ in } e$$
$$\mid \texttt{eq0 } e \mid \texttt{if } e \texttt{ then } e \texttt{ else } e$$

$$v ::= n \mid \texttt{true} \mid \texttt{false}$$

- In terms of interpreter definitions:

```
type exp =
  | Val of value
  | … (* as before *)
  | Eq0 of exp
  | If of exp * exp * exp
```

```
type value =
    Int of int
  | Bool of bool
```

Rules for Eq0 and Booleans

$$\frac{}{A;\ \mathtt{true} \Rightarrow \mathtt{true}}$$

$$\frac{A;\ e \Rightarrow 0}{A;\ \mathtt{eq0}\ e \Rightarrow \mathtt{true}}$$

$$\frac{}{A;\ \mathtt{false} \Rightarrow \mathtt{false}}$$

$$\frac{A;\ e \Rightarrow v \qquad v \neq 0}{A;\ \mathtt{eq0}\ e \Rightarrow \mathtt{false}}$$

- Booleans evaluate to themselves
  - A; **false** ⇒ **false**
- **eq0** tests for 0
  - A; **eq0 0** ⇒ **true**
  - A; **eq0 3+4** ⇒ **false**

Rules for Conditionals

$$\frac{A;\ \mathit{e1} \Rightarrow \texttt{true} \qquad A;\ \mathit{e2} \Rightarrow v}{A;\ \texttt{if}\ \mathit{e1}\ \texttt{then}\ \mathit{e2}\ \texttt{else}\ \mathit{e3} \Rightarrow v}$$

$$\frac{A;\ \mathit{e1} \Rightarrow \texttt{false} \qquad A;\ \mathit{e3} \Rightarrow v}{A;\ \texttt{if}\ \mathit{e1}\ \texttt{then}\ \mathit{e2}\ \texttt{else}\ \mathit{e3} \Rightarrow v}$$

- Notice that only one branch is evaluated
  - A; `if eq0 0 then 3 else 4` ⇒ `3`
  - A; `if eq0 1 then 3 else 4` ⇒ `4`

Quiz 3

What is the derivation of the following judgment?

`•; if eq0 3-2 then 5 else 10 ⇒ 10`

(a)

```
•; 3 ⇒ 3    •; 2 ⇒ 2    3-2 is 1
--------------------------
•; eq0 3-2  ⇒  false          •; 10 ⇒ 10
-------------------------------------
•; if eq0 3-2 then 5 else 10  ⇒  10
```

(b)

```
3 ⇒ 3   2 ⇒ 2
3-2 is 1
----------------
eq0 3-2 ⇒ false          10 ⇒ 10
-------------------------------
if eq0 3-2 then 5 else 10 ⇒ 10
```

(c)

```
•; 3 ⇒ 3
•; 2 ⇒ 2
3-2 is 1
----------
•; 3-2 ⇒ 1      1 ≠ 0
-------------------
•; eq0 3-2 ⇒ false       •; 10 ⇒ 10
-----------------------------------
•; if eq0 3-2 then 5 else 10 ⇒ 10
```

Quiz 3

What is the derivation of the following judgment?

$\bullet$; `if eq0 3-2 then 5 else 10` ⇒ `10`

(a)

```
•; 3 ⇒ 3    •; 2 ⇒ 2    3-2 is 1
-------------------------
•; eq0 3-2  ⇒  false        •; 10 ⇒ 10
-----------------------------------
•; if eq0 3-2 then 5 else 10  ⇒  10
```

(b)

```
3 ⇒ 3   2 ⇒ 2
3-2 is 1
----------------
eq0  3-2 ⇒ false          10 ⇒ 10
-------------------------------
if eq0 3-2 then 5 else 10 ⇒ 10
```

(c)

```
•; 3 ⇒ 3
•; 2 ⇒ 2
3-2 is 1
----------
•; 3-2 ⇒ 1     1 ≠ 0
------------------
•; eq0 3-2 ⇒ false      •; 10 ⇒ 10
----------------------------------
•; if eq0 3-2 then 5 else 10 ⇒ 10
```

Updating the Interpreter

```
let rec eval env e =
  match e with
    Ident x -> lookup env x
  | Val v -> v
  | Plus (e1,e2) ->
     let Int n1 = eval env e1 in
     let Int n2 = eval env e2 in
     let n3 = n1+n2 in
     Int n3
  | Let (x,e1,e2) ->
     let v1 = eval env e1 in
     let env' = extend env x v1 in
     let v2 = eval env' e2 in v2
  | Eq0 e1 ->
     let Int n = eval env e1 in
     if n=0 then Bool true else Bool false
  | If (e1,e2,e3) ->
     let Bool b = eval env e1 in
     if b then eval env e2
     else eval env e3
```

Basically both rules for **eq0** in this one snippet

Both **if** rules here

Quick Look: Type Checking

- Inference rules can also be used to specify a program's static semantics
  - I.e., the rules for type checking
- We won't cover this in depth in this course, but here is a flavor.

- Types t ::= **bool** | **int**
- Judgment $\vdash e : t$ says e has type t
  - We define inference rules for this judgment, just as with the operational semantics

Some Type Checking Rules

- Boolean constants have type **`bool`**

$$\frac{}{\vdash \texttt{true} : \texttt{bool}} \qquad \frac{}{\vdash \texttt{false} : \texttt{bool}}$$

- Equality checking has type **`bool`** too
  - Assuming its target expression has type **`int`**

$$\frac{\vdash e : \texttt{int}}{\vdash \texttt{eq0}\ e : \texttt{bool}}$$

- Conditionals

$$\frac{\vdash e1 : \texttt{bool} \qquad \vdash e2 : t \qquad \vdash e3 : t}{\vdash \texttt{if}\ e1\ \texttt{then}\ e2\ \texttt{else}\ e3 : t}$$

Handling Binding

- What about the types of variables?
  - Taking inspiration from the environment-style operational semantics, what could you do?

- Change judgment to be $G \vdash e : t$ which says e *has type* t *under type environment* G
  - G is a map from variables x to types t
    - Analogous to map A, but maps vars to types, not values

- What would be the rules for `let`, and variables?

Type Checking with Binding

- Variable lookup

$$\frac{G(\mathbf{x}) = t}{G \vdash \mathbf{x} : t}$$

analogous to

$$\frac{A(\mathbf{x}) = v}{A;\ \mathbf{x} \Rightarrow v}$$

- Let binding

$$\frac{G \vdash e1 : t1 \qquad G, x:t1 \vdash e2 : t2}{G \vdash \texttt{let } x = e1 \texttt{ in } e2 : t2}$$

analogous to

$$\frac{A;\ e1 \Rightarrow v1 \qquad A, x:v1;\ e2 \Rightarrow v2}{A;\ \texttt{let } x = e1 \texttt{ in } e2 \Rightarrow v2}$$

Scaling up

- Operational semantics (and similarly styled typing rules) can handle full languages
  - With records, recursive variant types, objects, first-class functions, and more
- Provides a concise notation for explaining what a language does. Clearly shows:
  - Evaluation order
  - Call-by-value vs. call-by-name
  - Static scoping vs. dynamic scoping
  - ... We may look at more of these later

Scaling Up: Lego City