CMSC 330
Organization of Programming Languages

OCaml
Higher Order Functions
Anonymous Functions

- Recall code blocks in Ruby
  \[(1..10).each \{ |x| print x \}\]
  - Here, we can think of \[\{ |x| print x \}\] as a function

- We can do this (and more) in OCaml
Anonymous Functions

- As with Ruby, passing around functions is common
  - So often we don’t want to bother to give them names

- Use `fun` to make a function with no name

```
(fun x -> x + 3) 5 = 8
```
Anonymous Functions

- **Syntax**
  - \( \text{fun } x_1 \ldots x_n \rightarrow e \)

- **Evaluation**
  - An anonymous function is an expression
  - In fact, *it is a value* – no further evaluation is possible
    - As such, it can be passed to other functions, returned from them, stored in a variable, etc.

- **Type checking**
  - \( (\text{fun } x_1 \ldots x_n \rightarrow e) : (t_1 \rightarrow \ldots \rightarrow t_n \rightarrow u) \)
    - when \( e : u \) under assumptions \( x_1 : t_1, \ldots, x_n : t_n \).
    - (Same rule as \( \text{let } f x_1 \ldots x_n = e \))
Quiz 1: What does this evaluate to?

\[
\text{let } y = (\text{fun } x \rightarrow x+1) \ 2 \ \text{in} \\
(\text{fun } z \rightarrow z-1) \ y
\]

A. Error
B. 2
C. 1
D. 0
Quiz 1: What does this evaluate to?

```ocaml
let y = (fun x -> x+1) 2 in
(fun z -> z-1) y
```

A. Error
B. 2
C. 1
D. 0
Quiz 2: What is this expression’s type?

(fun x y -> x) 2 3

A. Type error
B. int
C. int -> int -> int
D. 'a -> 'b -> 'a
Quiz 2: What is this expression’s type?

(fun x y -> x) 2 3

A. Type error
B. int
C. int -> int -> int
D. 'a -> 'b -> 'a
Functions and Binding

- Functions are **first-class**, so you can bind them to other names as you like
  
  ```
  let f x = x + 3;;
  let g = f;;
  g 5 = 8
  ```

- In fact, **let** for functions is a syntactic **shorthand**
  
  ```
  let f x = body
  ↓
  is semantically equivalent to
  let f = fun x -> body
  ```
Example Shorthands

- `let next x = x + 1`
  - Short for `let next = fun x -> x + 1`

- `let plus x y = x + y`
  - Short for `let plus = fun x y -> x + y`
Quiz 3: What does this evaluate to?

let f = fun x -> 0 in
let g = f in
let h = fun y -> g (y+1) in
h 1

A. 0
B. 1
C. 2
D. Error
Quiz 3: What does this evaluate to?

let f = fun x -> 0 in
let g = f in
let h = fun y -> g (y+1) in
h 1

A. 0
B. 1
C. 2
D. Error
Defining Functions Everywhere

let move l x =
    let left x = x - 1 in (* locally defined fun *)
    let right x = x + 1 in (* locally defined fun *)
    if l then left x
    else right x
;;

let move' l x = (* equivalent to the above *)
    if l then (fun y -> y - 1) x
    else (fun y -> y + 1) x
Pattern Matching With Fun

- match can be used within fun
  
  \[
  \text{(fun l \to \text{match } l \text{ with } (h::\_ ) \to h)} \ [1; 2] \ = \ 1
  \]

  • But use named functions for complicated matches

- May use standard pattern matching abbreviations, too

  \[
  \text{(fun (x, y) \to x+y)} \ (1, 2) \ = \ 3
  \]
Passing Functions as Arguments

In OCaml you can pass functions as arguments

```ocaml
let plus_three x = x + 3 (* int -> int *)

let twice f z = f (f z) (* ('a->'a) -> 'a -> 'a *)
```

twice's `f` parameter is a function

Calls the parameter function `f`

(twice!)

```ocaml
twice plus_three 5 = 11
```
map
The Map Function

OCaml’s `map` is a higher order function; like Ruby’s `collect`

- `map f l` takes a function `f` and a list `l`, applies function `f` to each element of `l`, and returns a list of the results (preserving order)

\[
\text{map } f \ [v_1; v_2; \ldots; v_n]
\]

\[=
[f \ v_1; f \ v_2; \ldots; f \ v_n]\]

```ocaml
let add_one x = x + 1
let negate x = -x

map add_one [1; 2; 3]  = [2; 3; 4]
map negate [9; -5; 0]  = [-9; 5; 0]
```
How can we implement Map?

```ocaml
let rec add1all l =  
    match l with  
        [] -> []  
    | h::t ->  
        (add_one h):: add1all t

let rec negall l =  
    match l with  
        [] -> []  
    | h::t ->  
        (neg h):: negall t

let rec map f l =  
    match l with  
        [] -> []  
    | h::t -> (f h)::(map f t)
```
Implementing map

```
let rec map f l =
    match l with
    [] -> []
  | h::t -> (f h)::(map f t)
```

What is the type of `map`?

```
( ) -> ( )
  f  l
```
Implementing map

What is the type of `map`?

\[
\text{let rec } \text{map } f \ l = \\
\quad \text{match } l \text{ with} \\
\quad \quad [] \rightarrow [] \\
\quad \quad | h::t \rightarrow (f \ h)::(\text{map } f \ t)
\]

\[
('a \rightarrow 'b) \rightarrow 'a \text{ list} \rightarrow 'b \text{ list}
\]

f

l
Another Example

Apply a list of functions to list of ints

```
let neg x = -x;;
let add_one x = x+1;;
let double x = x + x;;
let fs = [neg; add_one; double];;
let lst = [1;2;3];;
```

```
map (fun f -> map f lst) fs =
[[[-1; -2; -3]; [2; 3; 4]; [2; 4; 6]]

(neg 1) (neg 2) (neg 3) (add_one 1) ... (double 1) ...
```
map, as a cartoon

\[ \text{map cook} = \]

\[
\begin{bmatrix}
\text{cow}, & \text{cup}, & \text{chicken}, & \text{corn} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{hamburger}, & \text{fries}, & \text{chicken}, & \text{popcorn} \\
\end{bmatrix}
\]

map is included in the standard List module, i.e., as List.map
Quiz 4: What does this evaluate to?

map (fun x -> x * 4) [1;2;3]

A. [1.0; 2.0; 3.0]
B. [4.0; 8.0; 12.0]
C. Error
D. [4; 8; 12]
Quiz 4: What does this evaluate to?

`map (fun x -> x * 4) [1;2;3]`

A. [1.0; 2.0; 3.0]
B. [4.0; 8.0; 12.0]
C. Error
D. [4; 8; 12]
Quiz 5: Which function to use?

map ??? [1; 0; 3] = [true; false; true]

A. fun x -> true
B. fun x -> x = 0
C. fun x -> x != 0
D. fun x -> x = (x != 0)
Quiz 5: Which function to use?

map ??? [1; 0; 3] = [true; false; true]

A. fun x -> true
B. fun x -> x = 0
C. fun x -> x != 0
D. fun x -> x = (x != 0)

Note type error!
fold
(and foldr)
Two Recursive Functions

Sum a list of ints

let rec sum l =
  match l with
  | [] -> 0
  | h::t -> h + (sum t)

# sum [1;2;3;4];;
- : int = 10

Concatenate a list of strings

let rec concat l =
  match l with
  | [] -> ""
  | h::t -> h ^ (concat t)

# concat ["a";"b";"c"];;
- : string = "abc"
Notice Anything Similar?

Sum a list of ints

```ml
let rec sum l =
    match l with
    [] -> 0
  | h::t -> (+) h (sum t)
```

Concatenate a list of strings

```ml
let rec concat l =
    match l with
    [] -> ""
  | h::t -> (^) h (concat t)
```

- The structure of the two functions is the same!
- Only the parts in red differ
  - What to return for an empty list (0 or "")
  - What function to apply to \( h \) and the result of the recursive call (+ or ^)
- foldr abstracts these similarities using higher order functions

Note: (+) treats + as a prefix function, so \( 1+2 = (+) 1 \ 2 \)
The foldr Function

Sum a list of ints

\[
\text{let rec sum l =}
\text{match l with}
\text{[ ] -> 0}
\text{| h::t -> (+) h (sum t)}
\]

Concatenate a list of strings:

\[
\text{let rec concat l =}
\text{match l with}
\text{[ ] -> }
\text{| h::t -> (^) h (concat t)}
\]

\[
\text{let rec foldr f a l =}
\text{match l with}
\text{[ ] -> a}
\text{| h::t -> f h (foldr f a t)}
\]

\[
\text{let sum l = foldr (+) 0 l}
\text{let concat l = foldr (^) "" l}
\]
So, What is foldr?

- foldr is a function that
  - takes a function of two arguments, a final value, and a list
  - processes the list by applying the function to the head and the recursive application of the function to the rest of the list, returning the final value for the empty list

\[
\text{foldr } f \ v \ [v_1; v_2; \ldots; v_n] = \\
f \ v_1 \ (f \ v_2 \ (\ldots (f \ v_n \ v)\ldots))
\]

so \(\text{foldr add 0 [1;2;3;4]} = \)
\[
\text{add 1 (add 2 (add 3 (add 4 0)))} = 10
\]
Foldr and the Standard Library

- `List.fold_right` in the standard library is `foldr`, but with the order of its last two parameters reversed, i.e.,

  \[
  \text{fold_right } f \ [v_1; v_2; \ldots; v_n] \ v = f \ v_1 (f \ v_2 (\ldots(f \ v_n \ v))\ldots))
  \]

  so `fold_right add [1;2;3;4] 0 =
  
  \[
  \text{add 1 (add 2 (add 3 (add 4 0)))} = 10
  \]
Fold (aka fold_left)

- The List module also defines fold_left
  - which we will just call fold

```
let rec fold f a l =
  match l with
  [] -> a
| h::t -> fold f (f a h) t
```

- Similar to foldr, but changes the order of operations

```
let rec foldr f a l =
  match l with
  [] -> a
| h::t -> f h (foldr f a t)
```

Computes \( f \) on the accumulator \( a \) and the head \( h \), then passes the result as the accumulator to the recursive call.
What does \texttt{fold} do?

\begin{verbatim}
let rec fold f a l =
match l with
  [] -> a
| h::t -> fold f (f a h) t
\end{verbatim}

let add a x = a + x
fold add 0          \[1; 2; 3\] →
fold add (add 0 1)  \[2; 3\] →
fold add 1          \[2; 3\] →
fold add (add 1 2)  \[3\] →
fold add 3          \[3\] →
fold add (add 3 3)  [] →
fold add 6          [] →
6

We just built the \texttt{sum} function!
Fold (aka fold_left)

- What does fold do?

  - fold \( f \) \( v \) \([v_1; v_2; \ldots; v_n]\)
    
    \[= \text{fold} \ f \ (f \ v \ v_1) \ [v_2; \ldots; v_n] \]
    
    \[= \text{fold} \ f \ (f(f \ v \ v_1) \ v_2) \ [\ldots; v_n] \]
    
    \[= \ldots \]
    
    \[= f \ (f \ (f \ (f \ v \ v_1) \ v_2) \ \ldots) \ v_n \]

  - e.g., fold add 0 \([1;2;3;4]\) =
    
    add (add (add (add 0 1) 2) 3) 4 = 10

```
let rec fold f a l =
  match l with
  | [] -> a
  | h::t -> fold f (f a h) t
```
Another Example

```
let rec fold f a l =
  match l with
  | [] -> a
  | h::t -> fold f (f a h) t

let next a _ = a + 1
fold next 0          [2; 3; 4] →
fold next (next 0 2) [3; 4] →
fold next 1          [3; 4] →
fold next (next 1 3) [4] →
fold next 2          [4] →
fold next (next 2 4) [] →
fold next 3          [] →
3
```

We just built the `length` function!
Using Fold to Build Reverse

Let’s build the reverse function with fold!

```ocaml
let prepend a x = x::a
fold prepend [] [1; 2; 3; 4] →
fold prepend [1] [2; 3; 4] →
fold prepend [2; 1] [3; 4] →
fold prepend [3; 2; 1] [4] →
fold prepend [4; 3; 2; 1] [] →
[4; 3; 2; 1]
```

```ocaml
let rec fold f a l =
  match l with
  | [] -> a
  | h::t -> fold f (f a h) t
```

Quiz 6: What does this evaluate to?

```
let f x y = if x > y then x else y in
fold f 0 [3;4;2]
```

A. 0  
B. true  
C. 2  
D. 4
Quiz 6: What does this evaluate to?

```
let f x y = if x > y then x else y in
fold f 0 [3;4;2]
```

A. 0
B. true
C. 2
D. 4
Quiz 7: What does this evaluate to?

```
fold (fun a y -> a-y) 0 [3;4;2]
```

A. -9  
B. -1  
C. [2;4;3]  
D. 9
Quiz 7: What does this evaluate to?

fold (fun a y -> a-y) 0 [3;4;2]

A.  -9
B.  -1
C.  [2;4;3]
D.  9
Type of fold_left, fold_right

let rec fold_left f a l =  
  match l with  
  | [] -> a  
  | h::t -> fold_left f (f a h) t

('a -> 'b -> 'a) -> 'a -> 'b list -> 'a

f  a  l

let rec fold_right f l a =  
  match l with  
  | [] -> a  
  | h::t -> f h (fold_right f t a)

('b -> 'a -> 'a) -> 'b list -> 'a -> 'a

f  l  a
Summary: Left-to-right vs. right-to-left

fold_left \( f \ [v_1; v_2; \ldots; v_n] = \)
\[ f (f (f v v_1) v_2) \ldots v_n \]

fold_right \( f \ [v_1; v_2; \ldots; v_n] \ v = \)
\[ f v_1 (f v_2 (\ldots (f v_n v) \ldots)) \]

\[
\text{fold_left (fun x y -> x - y) 0 [1;2;3] = -6 since ((0-1)-2)-3) = -6}
\]

\[
\text{fold_right [1;2;3] (fun x y -> x - y) 0 = 2 since 1-(2-(3-0)) = 2}
\]
When to use one or the other?

- Many problems lend themselves to `fold_right`
- But it does present a performance disadvantage
  - The recursion builds of a deep stack: One stack frame for each recursive call of `fold_right`
- An optimization called `tail recursion` permits optimizing `fold_left` so that it uses no stack at all
  - We will see how this works in a later lecture!
Combining map and fold

- Idea: map a list to another list, and then fold over it to compute the final result
  - Basis of the famous “map/reduce” framework from Google, since these operations can be parallelized

```plaintext
let countone l = fold (fun a h -> if h=1 then a+1 else a) 0 l
let countones ss = let counts = map countone ss in fold (fun a c -> a+c) 0 counts

countones [[1;0;1]; [0;0]; [1;1]] = 4
countones [[1;0]; []; [0;0]; [1]] = 2
```
fold & map

More examples, practice
Map Example 1: Permute a list

```
let permute lst =
    let rec rm x l = List.filter ((<>) x) l
    and insertToPermute lst x =
        let t = rm x lst in
        List.map ((fun a b->a::b) x) (permuteall t)
    and permuteall lst =
        match lst with
        |[]->[[]]
        |[x]->[[x]]
        |_->List.flatten(List.map (insertToPermute lst) lst)
    in permuteall lst

;;

# permute [1;2;3];;
- : int list list =
[[1; 2; 3]; [1; 3; 2]; [2; 1; 3]; [2; 3; 1]; [3; 1; 2]; [3; 2; 1]]
```
Map Example 2: Power Set

```
let populate a b =  
  if b=[] then [[a]]  
  else let t = List.map (fun x->a::x) b in  
         [a]::t@b

let powerset lst = List.fold_right populate lst []

# populate 1 [[2];[3]];;  
- : int list list =  
  [[1]; [1; 2]; [1; 3]; [2]; [3]]

# powerset [1;2;3];;  
- : int list list = [[1]; [1; 2]; [1; 2; 3]; [1; 3]; [2]; [2; 3]; [3]]
```
Fold Example 1: Product of an int list

let mul x y = x * y;;

let lst = [1; 2; 3; 4; 5];;;

fold mul 1 lst
- : int = 120

Wrong accumulator

fold mul 0 lst;;
- : int = 0
Fold Example 2: Count elements of a list satisfying a condition

```
let countif p l =
fold (fun counter element -> if p element then counter+1
else counter) 0 l ;;

countif (fun x -> x > 0) [30;-1;45;100;0];;
- : int = 3
```
Fold Example 3: Collect even numbers in the list

let f acc y = if (y mod 2) = 0 then y::acc
else acc;;

fold f [] [1;2;3;4;5;6];;

- : int list = [6; 4; 2]
Fold Example 4: Inner Product

first compute list of pair-wise products, then sum up

\[ [x_1;x_2;x_3] \times [y_1;y_2;y_3] = x_1 \times y_1 + x_2 \times y_2 + x_3 \times y_3 \]

let rec map2 f a b =
    match (a,b) with
    |([],[]) -> ([])
    |(h1::t1,h2::t2) -> (f h1 h2):: (map2 f t1 t2)
    |_ -> invalid_arg "map2";;

let product v1 v2 =
    fold (+) 0 (map2 (\*\) v1 v2);;
# val product : int list -> int list -> int = <fun>
product [2;4;6] [1;3;5];;
#- : int = 44
Fold Example 5: Find the maximum from a list

let maxList lst =
    match lst with
    [] -> failwith "empty list"
    | h::t -> fold max h t ;;

maxList [3;10;5];;
- : int = 10

(*
maxList [3;10;5]
fold max 3 [10;5]
fold max (max 3 10) [5]
fold max (max 10 5) []
fold max 10 []
10 *)
Quiz: Sum of sublists

Given a list of int lists, compute the sum of each int list, and return them as list.

For example:

sumList [[1;2;3];[4];[5;6;7]]
- : int list = [6; 4; 18]
Solution: Sum of sublists

let sumList = map (fold (+) 0 );;

sumList [[1;2;3];[4;5;6];[10]];;
- : int list = [6; 15; 10]
Quiz: Maximum contiguous sublist

Given an int list, find the contiguous sublist, which has the largest sum and return its sum.

Example:
Input: [-2,1,-3,4,-1,2,1,-5,4]
Output: 6
Explanation: [4,-1,2,1] has the largest sum = 6