The geometric Set k-cover problem through the prism of polychromatic colorings of range spaces and planar graphs

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Abstract

In this paper, we review the Set k-cover problem on certain geometric graphs as applied to energy efficient monitoring in wireless sensor networks. Two approaches, originally applied to other problems, are discussed. One of them is Aloupis *et al.*'s [1] where geometric range spaces are colored with slightly more than k colors so that every range of size at least k has k or more colors in it. The second one is Alon *et al.*'s [2] where it is shown that every planar graph whose faces have at least g vertices on their boundary can be colored with $\lfloor(3g-5)/4\rfloor$ colors so that every face has every color appearing on its boundary. We discuss the immediate results that these approaches give to the Set K-Cover problem, a roughly $\frac{1}{24}$-approximation algorithm and a sufficient condition for a very strong solution of the Set k-cover, namely, if every target is covered by at least g sensors and for every subset of sensors S' and targets T' the number of incidences (incidence is a sensor monitoring a target) is bounded by $2(|S'|+|T'|)$ then the set of sensors can be partitioned in roughly $\frac{3g}{4}$ groups so that every target is monitored by a sensor in every group. These immediate applications are not satisfactory solutions to the Set k-cover problem. However, it is not clear how to extend the approaches discussed to fully solve it.

1 Introduction

Consider the Set k-cover problem. A set of n sensors monitors a set of m targets. The problem can be modeled with a bipartite graph $G=(S,T,E)$ where the sensors are denoted by S and the targets are denoted by T. There is an edge between sensor $s_i \in S$ and target $t_j \in T$ if s_i monitors t_j. Our goal is to extend the battery life of the sensor network as long as possible. Assume that the battery of a sensor lasts for 1 time unit. Then, if we turn all sensors on, the network will be out of energy after 1 unit of time. Instead, we want to partition the sensors in k disjoint groups and then activate the sensors within a group only, alternating the groups in a round robin fashion. Thus, we will extend the battery life of the network k-fold.

Ideally, the sensors in each group will cover all of the targets. However, this might not always be possible. For example, consider a target t that is monitored by two sensors only. Then, it is certainly infeasible to partition the set of sensors into three or more time slots so that t is monitored in each of them. Yet, it might still be acceptable if a target, even though not monitored in all time slots, is monitored in a lot of them. For this reason, we cast the partitioning problem as an optimization problem.

Before describing the optimization function, we note an alternative, sometimes more convenient, way to model the problem. Consider a hypergraph $H=(S,E)$ where the set of nodes represents the set of sensors in the network. Then, each target is represented as the hyperedge of nodes that monitor this target. We now want to color the nodes of this hypergraph with k colors which will correspond to partitioning the nodes of the graph in k groups. The benefit of a hyperedge e, $b(e)$, is defined as the number of colors represented among the nodes in e. The optimization function that we then want to maximize is as follows: $\sum_{e\in E} b(e)$. In other words, we are maximizing the number of slots in which the targets are monitored. Note that this maximization is not necessarily fair, i.e. a target may be monitored in a very small number of the time slots whereas another target is monitored in most of the time slots. A fairness requirement would be that every target is monitored in a certain fraction of the time slots.

The Set k-cover model was first proposed in [3]. The problem as defined above is NP-hard even in the case when a target is monitored by two sensors only. In this case, the hypergraph H becomes a classical graph and the problem becomes equivalent to the Max K-Cut problem in which we want to partition the set of nodes in k groups so that the number of edges with

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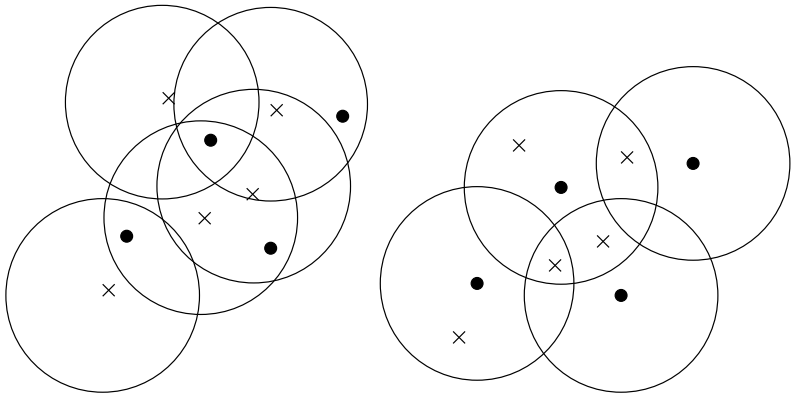

(a) Disks are drawn around sensors. The sensors cover those targets that are within their disk.

(b) Disks are drawn around targets. A target is covered by the sensors within its disk. The sensors within a disk form a hyperedge in H.

Figure 1: Two ways to look at the geometric Set k-cover problem. Sensors are represented with a cross and targets are represented with a dot.

endpoints in different groups is maximized. This problem is known to be NP-hard [4].

Abrams *et. al.* [5] propose three approximation algorithms for the Set k-cover problem. The first algorithm is a randomized algorithm. It assigns independently to each sensor a random color from $\{1, ..., k\}$. Abrams *et al.* show that in expectation this algorithm does better than $1 - \frac{1}{e}$ times the optimum. The other two algorithms are a $1/2$-approximation and a $(1 - \frac{1}{e})$-approximation deterministic greedy algorithms where the last one is the derandomization of the randomized algorithm.

The randomized algorithm is a very simple and very effective algorithm and is a baseline to compare against other algorithms. In [6], Deshpande *et al.* propose the hypergraph model and an algorithm that performs better than the $1 - \frac{1}{e}$ ratio when the size of the hyperedges is not too big, no more than three. In their approach, they replace each hyperedge with a clique, thus transforming the hypergraph into a graph on which they apply the semidefinite programming based approximation algorithm for Max K-Cut from [7]. However, the algorithm does not perform well when the size of the hyperedges can be big.

Without further structure in the bipartite graph of the sensor network it seems difficult to improve upon the randomized algorithm. For this reason we consider the Set k-cover problem in the following geometric setting. Sensors and targets are modeled as points on the plane. In addition, a sensor can monitor targets within distance 1 from the sensor. In other words, a sensor covers the area of a unit disk around the sensor (all sensors are uniform). Equivalently, disks of radius 1 centered around the targets give all the sensors that monitor the target, and thus, these disks represent the hyperedges in the modeling hypergraph H (see Figure 1). Note that Figure 1(a) can be interpreted as a hypergraph as well. It is the dual (for definitions see Section 2) of the hypergraph on the targets whose hyperedges are those sets of targets that fall within a sensor disk.

The rest of the paper is organized as follows. In Section 2, we review the problem and solution proposed by Aloupis *et al.* in [1] which gives roughly a $\frac{1}{24}$-approximation to the Set k-cover problem. In Section 3, we review the problem and solution proposed by Alon *et al.* in [2]. In each section we discuss the immediate applications to the Set k-cover problem. Finally, in Section 4 we conclude.

2 Coloring geometric range spaces

In [1], Aloupis *et al.* look at the following problem. Given a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) how many colors do you need to color the points with so that every region of a certain family (we will be interested in disks) and certain size has many colors. This is precisely the type of question that we are interested in. Our family of regions is the targets with their corresponding

unit radius disks. Each such disk is a hyperedge in the modeling hypergraph H and it is desirable that it has as many colors as possible.

2.1 Definitions and problem formulation

We start with a few definitions.

Definition 2.1. **([1])** *A range space (also a hypergraph) is a pair $\mathcal{S} = (S, R)$ where S is a set and R is a set of subsets of S. $\mathcal{D} = (\mathbb{R}^2, R)$ is the geometric range space of disks if R consists of all disks in the plane. $\mathcal{H} = (\mathbb{R}^2, R)$ is the geometric range space of halfplanes if R is the set of all halfplanes. $\mathcal{L} = (\mathbb{R}^3, R)$ is the geometric range space of lower halfspaces if R is the set of all lower halfspaces.*

The geometric range spaces $\mathcal{D}, \mathcal{H}, \mathcal{L}$ are infinite. We define a finite restriction of a geometric range space.

Definition 2.2. **([1])** *A finite restriction of the geometric range space $\mathcal{S} = (\mathbb{R}^d, R)$ is the range space $\mathcal{F} = (V, E)$ where V is a finite set of points in $\mathbb{R}^d$ and $E = \{e | e \subseteq V, e = V \cap r, r \in R\}$.*

Next, we define the dual of a range space.

Definition 2.3. **([1])** *Let $\mathcal{S} = (S, R)$ be a range space. For every $p \in S$ define $r(p) = \{r | r \in R, r \ni p\}$. The dual of $\mathcal{S}$ is the range space $\widetilde{\mathcal{S}} = (R, T)$ where $T = \{r(p) | p \in S\}$.*

A c-coloring of the range space $\mathcal{S} = (S, R)$ is assignment of at most c colors to the elements of S. A range $r \in R$ is k-colorful if it contains at least k elements of S of different colors. Aloupis *et al.* bound the following two functions:

- $c_{\mathcal{S}}(k)$ denotes the minimum number c such that there exists a c-coloring of any finite restriction of $\mathcal{S}$ such that every range r is $\min(|r|, k)$-colorful.
- $p_{\mathcal{S}}(k)$ denotes the minimum number p such that there exists a k-coloring of any finite restriction of $\mathcal{S}$ such that every range of size at least p is k-colorful.

The motivation of Aloupis *et al.* is that it is not always possible to color a finite range space with k colors so that every range is $\min(|r|, k)$-colorful. For example, every graph with an odd cycle cannot be 2-colored in such a way. However, if one is allowed to use a little bit more colors than k, maybe one can color the range space so that every range is $\min(|r|, k)$-colorful. For example, one can color every graph of n vertices with n colors so that the above restriction is satisfied. The goal though is to bound the number of colors in terms of k only.

For the case of $p_{\mathcal{S}}(k)$, let us call that $s \in S$ covers $r \in R$ if $s \in r$. Then $p_{\mathcal{S}}(k)$ denotes the minimum number of times a range needs to be covered so that every finite restriction S can be partitioned into k groups in a way such that every range of the finite restriction that is covered at least $p_{\mathcal{S}}(k)$ times is covered by an element of every group. Clearly, it may not be sufficient to cover a range k times. This question is related to our sensor covering problem in the following sense. What is the minimum number of sensors that need to cover a target so that we can partition the sensors into k groups (equivalently k-color the sensors) so that every target is covered by sensors in all k groups (equivalently, is monitored by sensors in all k time slots).

Aloupis *et al.* bound some of $c_{\mathcal{S}}(k)$, $c_{\widetilde{\mathcal{S}}}(k)$, $p_{\mathcal{S}}(k)$, and $p_{\widetilde{\mathcal{S}}}(k)$ for several range spaces such as halfplanes, lower halfspaces, translates of a centrally symmetric convex polygon, disks and pseudo-disks.

For example, the following theorem is shown to hold for lower halfspaces.

Theorem 2.4. **([1])** *$c_{\mathcal{L}}(k) = O(k)$. In other words, there exists a constant c s.t. one can color the points of every finite restriction of $\mathcal{L}$ with at most ck colors so that every halfspace h is $\min(|h|, k)$-colorful.*

We are interested in the range space on disks $\mathcal{D}$ since this will give us an approximation to the geometric Set k-cover problem. Using this theorem, it is easy to show the following corollary.

Corollary 2.5. **([1])** *$c_{\mathcal{D}}(k) = O(k)$.*

Proof. Consider a finite restriction of $\mathcal{D}$. Lift the points on the plane orthogonally to the paraboloid $z = x^2 + y^2$. Then, disks lift to the set of points that is cut by a plane intersecting the paraboloid. The points that are inside the disk lift to the part of the paraboloid below this plane on the paraboloid and points outside of the disk lift to the part of the paraboloid above the plane. By Theorem 2.4 we can always color the lifted points with $O(k)$ colors so that every lowerspace l is $\min(|l|, k)$-colorful. Using the same colors for the points on the plane, every disk d is $\min(|d|, k)$-colorful. □

The constant in the above corollary hidden in the O notation is somewhat big. Using a more direct approach we can obtain a lower constant. The following theorem is shown in [1]

Theorem 2.6. **([1])** *$c_{\widetilde{\mathcal{D}}}(k) \leq 24k + 1$*

Let $\mathcal{U}$ denote the range space $(\mathbb{R}^2, R)$ where R is the set of disks of fixed size, say unit disks. It is easy to see the following

Lemma 2.7. *$c_{\mathcal{U}}(k) = c_{\widetilde{\mathcal{U}}}(k)$.*

Proof. Assume there exists a valid coloring of every finite restriction of $\mathcal{U}$ with $c(k)$ colors. Let (T, E) be a finite restriction of $\widetilde{\mathcal{U}}$. Consider the finite restriction of $\mathcal{U}, (C(T), D)$ defined by the finite set $C(T)$ which denotes the centers of the disks in T. It has a valid $c(k)$-coloring since it is a finite restriction of $\mathcal{U}$. Now, color an element $t \in T$ the way its center was colored in this valid coloring. We claim that this coloring is valid for (T, E). Consider a hyperedge $e \in E$. There exists $s \in \mathbb{R}^2$ s.t. $e = e(s) = \{t | t \in T, s \in t\}$. It follows that the unit disk d centered at s contains the centerpoints of the disks in $e(s)$ and only them. We know that this disk is $\min(|e|, k)$-colorful. Thus, e is $\min(|e|, k)$-colorful and the lemma follows. □

We obtain

Corollary 2.8. *$c_{\mathcal{U}}(k) \leq 24k + 1$*

However, the same bound can be shown directly on $c_{\mathcal{D}}(k)$ essentially using the same argument as in Theorem 2.6. We show this argument in the next subsection.

With regard to the $p_{\mathcal{D}}(k)$ function, Aloupis *et al.* [1] note that by a result of Pach *et al.* [8] $p_{\mathcal{D}}(k) = \infty$. However, this result is for the range space $\mathcal{D}$ of all open disks, i.e. without their boundary and not unit size. Aloupis *et al.* give no bound on $p_{\widetilde{\mathcal{D}}}(k)$.

2.2 Bounding $c_{\mathcal{D}}(k)$ directly

In the following let (S, D) denote a finite restriction of $\mathcal{D}$.

Definition 2.9. *Denote with $G_k(S)$ the graph with vertex set S and an edge between two points $p \in S$ and $q \in S$ if there is a disk $d \in D$ that contains at most k points other than p and q.*

Lemma 2.10. *$G_0(S)$ has at most $3n$ edges where $n = |S|$.*

Proof. $G_0(S)$ consists of the pairs of points in $p, q \in S$ for which there exists a disk $d \in D$ that contains only them. But this means that (p, q) is an edge in the Delaunay graph of the pointset S. Thus, $|G_0(S)|$ is bounded from above by the number of edges in the Delaunay graph G of S. Since G is a planar graph, by Euler's formula it follows this number is at most $3n - 6$. □

Definition 2.11. *The intersection graph of (S, D) is a graph on S with an edge between $p, q \in S$ if there is a disk in D that contains both p and q.*

Lemma 2.12. ([1]) *Let $G = (S, E)$ be a subgraph of the intersection graph of (S, D). For each edge e, choose a disk d_e that contains the endpoints of e. Let X denote the set of configurations (e, t) where $e = (p, q), t \in S \setminus \{p, q\}, t \in d_e$. Then, X is large. More precisely, suppose that $|E| \geq 6n$ where $|S| = n$. Then, $|X| \geq \frac{|E|^2}{12n}$.*

Proof. We first prove the bootstrapping inequality $|X| \geq |E| - 3n$ which is then used in the random sampling argument of the Crossing Lemma [9].

The proof of the inequality $|X| \geq |E| - 3n$ proceeds by induction on $|E| - 3n$. The claim is obvious if $|E| - 3n \leq 0$. Assume it holds for a non-negative $|E| - 3n = k$. We show it holds when $|E| - 3n = k + 1 > 0$.

First, note that X is nonempty for otherwise G is a subset of $G_0(S)$ and by Lemma 2.10 it follows $|E| \leq 3n$ contradicting $|E| - 3n = k + 1 > 0$. Pick a pair $(e, t) \in X$ and remove e from E to obtain E'. Let X' denote the subset of configurations induced by the graph $G' = (S, E')$. We have $|X| \geq |X'| + 1$ (at least (e, t) is removed from X). Furthermore, $|E'| - 3n = k$ and by the inductive hypothesis we have $|X'| \geq |E'| - 3n$. It follows, $|X| \geq |E| - 3n$.

We continue with the random sampling argument. Pick a random subset $S' \subseteq S$ where each point $p \in S$ is chosen to be included in S' with some probability a to be determined later. Denote with E' the set of edges in E whose endpoints are among S'. Denote further $n' = |S'|$ and $m' = |E'|$. Finally, let X' denote the subset of configurations induced by S'. By the above inequality, we have $|X'| \geq m' - 3n'$. $|X'|, m', n'$ are random variables and for their expectation we get $\mathbf{E}[|X'|] \geq \mathbf{E}[m'] - 3\mathbf{E}[n']$. For the expectations we have $\mathbf{E}[n'] = an$, $\mathbf{E}[m'] = a^2|E|$ since both endpoints of an edge $e \in E$ must be in S' for the edge to be in E', and $\mathbf{E}[|X'|] = a^3|X|$ since a configuration $(e, t) \in X$ is in X' if both endpoints of e as well as t are in S'. It follows

$$a^3|X| \geq a^2|E| - 3an$$

If we set $a = \frac{6n}{|E|} \leq 1$ we get the assertion of the lemma $|X| \geq \frac{|E|^2}{12n}$. □

Definition 2.13. *A graph G is called k-degenerate if every subgraph of G has a vertex that is of degree at most k.*

Lemma 2.14. *$G_k(S)$ is $24k$-degenerate.*

Proof. Let $G = (S', E')$ be a subgraph of $G_k(S)$ which in turn is a subgraph of the intersection graph of (S, D). Denote with X' a set of configurations for this graph. By Lemma 2.12 we have that $|X'| \geq \frac{|E'|^2}{12n'}$ where $n' = |S'|$. It also holds $|X'| \leq k|E'|$. Combining the two inequalities it follows $|E'| \leq 12kn'$. Thus, the sum of the degrees of the vertices in G, which is $2|E'|$ is at most $24kn'$ which, in turn, implies the existence of a vertex $p \in S'$ of degree at most $24k$ We conclude $G_k(S)$ is $24k$-degenerate. (Note that we can apply Lemma 2.12 if $|E'| \geq 6n'$, but if this is not the case the graph then it trivially follows $|E'| < 6n' < 12n'$.) □

Theorem 2.15. $c_{\mathcal{D}}(k) \leq 24k + 1$

Proof. Call a coloring of a finite restriction (S, D) of $\mathcal{D}$ valid if every disk $d \in D$ is $\min(|d|, k)$-colorful. We show how to color an arbitrary finite restriction of $\mathcal{D}$ with $24k + 1$ colors in a valid way.

Consider the graph $G_k(S)$ and pick a point $p \in S$ that has degree at most $24k$ in $G_k(S)$. Color the rest of the points inductively. Now consider a disk d. If it does not contain p then it is already $\min(|d|, k)$ colorful by the inductive hypothesis. This holds true for disks that contain p as well as k other points of S. Consider the situation when d contains at most $k-1$ other points besides p. By the inductive hypothesis, they are all of different colors. Note also that all of them are neighbors of p in $G_k(S)$. Thus, over all disks that contain p together with at most $k - 1$ other points there are at most $24k$ such neighbors of p in $G_k(S)$. It follows that there exists a color different from p's neighbors colors and coloring p with this colors retains the validity of the coloring.

We note that the above proof is algorithmic. □

2.3 Application to the Geometric Set k-cover problem

The following roughly $\frac{1}{24}$ approximation is an immediate application of the above result.

Corollary 2.16. *There exists a* $(\frac{1}{24} - \frac{25}{24k})$*-approximation algorithm to the Geometric Set* k*-cover problem.*

Proof. By Theorem 2.15, the sensors can be colored with k colors so that every disk d is $\min(|d|, \lfloor \frac{k-1}{24} \rfloor)$-colorful. A disk d is optimally colored if it is $\min(|d|, k)$-colorful. Since $\frac{\min(|d|, \lfloor \frac{k-1}{24} \rfloor)}{\min(|d|, k)} > \frac{\frac{k-1}{24} - 1}{k} = \frac{1}{24} - \frac{25}{24k}$, this coloring is $(\frac{1}{24} - \frac{25}{24k})$-approximation to the Geometric Set k-cover problem. □

3 Polychromatic colorings

Alon *et al.* [2] solve the following problem. Let G be a planar graph. For a face $f \in G$, denote with $size(f)$ the number of vertices on the boundary of f. Then, one can always color the vertices of G with $\lfloor \frac{3g-5}{4} \rfloor$ colors so that every color appears on the boundary of G. In the following, we formulate and prove this theorem in the more general setting of hypergraphs.

3.1 A sufficient condition for lowerbounding the polychromatic number of a hypergraph

We first define the polychromatic number of a hypergraph.

Definition 3.1. *A coloring of the vertices of a hypergraph H is called polychromatic if every color appears in every hyperedge of H. The largest number k such that there exists a polychromatic k-coloring of H is called the polychromatic number $p(H)$ of the hypergraph H.*

In this section we prove the following theorem giving a sufficient condition to lowerbound the polychromatic number of a hypergraph.

Theorem 3.2. ([2]) *Let $H = (V, E)$ be a hypergraph. If for every $V' \subseteq V$ and $E' \subseteq E$, the number of incidences between nodes in V' and edges in $E', i(V', E') \leq 2(|V'| + |E'|)$ then if each edge of H is of cardinality at least g, $p(H) \geq \lfloor \frac{3g-5}{4} \rfloor$.*

Proof. The proof proceeds in two steps. First, vertices are assigned to hyperedges in such a way that no vertex gets assigned more than twice. In the second step, a graph derived from this assignment is appropriately edge-colored. This coloring will give a polychromatic coloring of the original hypergraph. We proceed using several lemmas.

Lemma 3.3. *Vertices can be assigned to hyperedges in such a way that every hyperedge has $g-2$ of its vertices assigned to it while no vertex gets assigned more than twice.*

Proof. Consider the flow network that consists of a bipartite graph on partitions E and V and a source s and a sink t. A node corresponding to a hyperedge $e \in E$ is connected to a node corresponding to a vertex $v \in V$ with capacity 1 if $v \in E$ and 0 otherwise. Denote this capacity with a_{ev}. In addition, the source s is connected to all nodes $e \in E$ with capacity $g - 2$ and all nodes $v \in V$ are connected to the sink t with capacity 2. Then, the min cut of this network (equivalently the

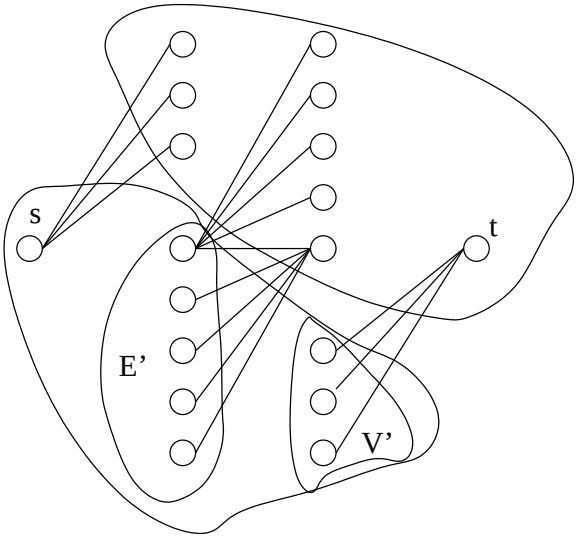

Figure 2: Illustration of the proof of the assignment lemma

max flow) is $(g-2)|E|$ if and only if for every subset $E' \subseteq E$ and $V' \subseteq V$, the cut comprised by $\{s\} \cup E' \cup V'$ and its complement has capacity at least $(g-2)|E|$ (see Figure 2)(note that the cut $(\{s\}, V \setminus \{s\})$ achieves capacity $(g-2)|E|$). In other words, the following must hold:

$$\sum_{\substack{e \in E' \\ v \in V \setminus V'}} a_{ev} + (|E| - |E'|)(g-2) + 2|V'| \geq (g-2)|E| \Leftrightarrow$$

$$\sum_{\substack{e \in E' \\ v \in V \setminus V'}} a_{ev} \geq (g-2)|E'| - 2|V'| \Leftrightarrow$$

$$\sum_{\substack{e \in E' \\ v \in V}} a_{ev} - \sum_{\substack{e \in E' \\ v \in V'}} a_{ev} \geq (g-2)|E'| - 2|V'| \Leftrightarrow$$

$$i(E', V') \leq \sum_{\substack{e \in E' \\ v \in V}} a_{ev} - (g-2)|E'| + 2|V'|$$

Since every hyperedge is of cardinality at least g it follows

$$\sum_{\substack{e \in E' \\ v \in V}} a_{ev} - (g-2)|E'| + 2|V'| \geq$$

$$g|E'| - (g-2)|E'| + 2|V'| =$$

$$2(|E'| + |V'|)$$

which together with $i(E', V') \leq 2(|V'| + |E'|)$ concludes the lemma. □

Given the assignment in Lemma 3.3 we construct the assignment graph G on the vertex set $E \cup \{s, t\}$ where $s, t \notin E$. For every vertex $v \in V$ create a v-edge in G in the following way. If v was assigned to two hyperedges e_1 and e_2, create the edge (e_1, e_2). If it was assigned to only one hyperedge e, create the edge (e, s). If it was assigned to no hyperedges, create (s, t). Note that G is loopless, undirected multigraph. We proceed with the following

Lemma 3.4. *The assignment graph G has a spanning bipartite graph B, i.e. B is a bipartite graph on the same vertex set as G, such that the degree of a vertex v in B, $d_B(v)$, is at least half the degree of v in G, $d_G(v)$, i.e. $d_B(v) \geq \lceil \frac{d_G(v)}{2} \rceil$*

Proof. Consider an arbitrary bipartite graph spanning G. If there is a vertex v s.t. $d_B(v) < \lceil \frac{d_G(v)}{2} \rceil$, move v in the other partition of B. By doing this, the number of edges going across the cut induced by B increases. Continue doing this. Since the size of the cut induced by B cannot increase infinitely, it is bounded at least by the max-cut of G, at some point we will stop. At this point, the condition of the lemma is satisfied. □

Denote the two partitions of the graph B from Lemma 3.4 with L and R. Next, we continue with the following orientation lemma

Lemma 3.5. *Let G' be the subgraph of the assignment graph G induced by either L or R. One can orient the edges of G' in such a way that the outgoing degree of a vertex in G', $d^+_{G'}(v)$ is at least $\lfloor \frac{d_{G'}(v)}{2} \rfloor$ where $d_{G'}(v)$ is the degree of v in G'.*

Proof. Consider the odd-degree vertices of G'. There is an even number of them. Add a perfect matching on them to the graph G'. This will make the degree of every vertex in G' even, and thus, the graph G' Euler. Consider an Euler tour of G'. It enters a fixed vertex v as many times as it exits it using each edge exactly once. Since at most one of these edges did not belong to G' originally, orienting the edges in the direction of the tour gives the claim of the lemma. □

Note that coloring the edges of the assignment graph G polychromatically corresponds to a polychromatic coloring of the vertices of the original hypergraph H. This is what we will set out to do next.

Lemma 3.6. *For every bipartite graph B, it is possible to color its edges with k colors so that for every vertex v in B and every color i, the number of edges incident with v of color i is either $\lfloor \frac{d_B(v)}{k} \rfloor$ or $\lceil \frac{d_B(v)}{k} \rceil$.*

Proof. Consider the graph B' obtained from B in the following way. If a vertex v has degree more than k split

v into $\lceil \frac{d(v)}{k} \rceil$ copies, assigning to each copy, except possibly to the last one, k of the edges incident with v in B so that no edge is assigned to two copies. The resulting graph has degree at most k. By König's Theorem [10], B' can be k-edge-colored so that no two edges incident with the same vertex have the same color. Reversing the transformation that produced B' from B gives the desired in the lemma edge-coloring of B. □

It follows from the assignment lemma that every vertex v in the assignment graph G corresponding to a hyperedge in H has degree at least $g-2$. If this is not the case for s or t, add enough edges between them to make the degree of each of them bigger than $g-2$. Thus, the minimum degree of G is at least $g-2$. The following lemma concludes the proof of Theorem 3.2.

Lemma 3.7. *The edges of the assignment graph G can be colored with $k = \lfloor \frac{3g-5}{4} \rfloor$ so that every vertex v of G is polychromatic, i.e. every color appears among the edges incident with v.*

Proof. Let B be the spanning bipartite graph of G from Lemma 3.4. Color the edges of B according to Lemma 3.6. Given a vertex v in B, after this step, there are $\min(d_B(v), k)$ colors among the edges incident with v in B. If not all colors are represented among these edges, i.e. $d_B(v) < k$, all the edges have different colors. Consider the orientation of the subgraph G' of G induced by v's partition from Lemma 3.5. Color the outgoing edges of v with new colors until one runs out of colors or edges. Then, the number of colors that the edges incident with v in G get is $\min(k, \lfloor \frac{d_G(v)-d_B(v)}{2} \rfloor + d_B(v))$. If we let $d = g-2$ be the minimum degree, we have $\lfloor \frac{d_G(v)-d_B(v)}{2} \rfloor + d_B(v) \geq \lceil \frac{d}{2} \rceil + \lfloor \frac{\lfloor \frac{d}{2} \rfloor}{2} \rfloor = \lfloor \frac{3d+1}{4} \rfloor = \lfloor \frac{3g-5}{4} \rfloor$. □

Every hyperedge e of H has a corresponding vertex in G. The edges incident with this vertex in G correspond to nodes of H that belong to e. Since all colors are represented among the edges, the hyperedge e is polychromatic. □

3.2 Polychromatic colorings of planar graphs

In this section we show how to use Theorem 3.2 to obtain a polychromatic coloring of a planar graph which is the main result of [2]. Given a planar graph G, consider the hypergraph $H = (V, E)$ where V is the same vertex set as the vertex set for G and E is constructed as follows. For every face f of G create a hyperedge $e \in E$ s.t. e contains the vertices on the boundary of f. Then every polychromatic coloring of H is a polychromatic coloring of G in the sense that every color will appear on the boundary of every face of G. In order to use Theorem 3.2 we have to show a bound on the number of incidences between an arbitrary subset of faces F' and arbitrary subset of vertices V'.

Lemma 3.8. ([2]) *Let $G = (V, E)$ be a planar graph. Denote the set of its faces with F. Let $V' \subseteq V$ and $F' \subseteq F$. Then the number of incidences, $i(F', V')$, between elements of V' and F' is bounded as follows: $i(F', V') \leq 2(|F'| + |V'|) - 3$.*

Proof. Consider the graph B on the vertex set $F' \cup V'$ where there is an edge between $f \in F'$ and $v \in V'$ if they are incident. The number of edges of B is exactly $i(F', V')$. This graph is simple, planar (consider putting a point inside each face of G and connecting it with every vertex on the boundary; this gives a planar embedding of B) and bipartite (the two partitions being V' and F'). Because the graph is bipartite, it follows it does not contain a triangle (it contains no odd cycles). Let n denote the number of vertices of B, i.e. $n = |V'| + |F'|$ and m be the number of faces in the planar embedding of B. If $n \geq 4$, we have $4m \leq 2e$ since B is triangle free. From this and Euler's formula $n - i(F', V') + m = 2$ it follows $i(F', V') \leq 2n - 4$, i.e. $i(F', V') \leq 2(|F'| + |V'|) - 4$. It is easy to check that when $n \leq 3$, $i(F', V') \leq 2(|F'| + |V'|) - 3$ □

Using this lemma and Theorem 3.2 we obtain

Theorem 3.9. ([2]) *Every planar graph whose faces are of size at most g can be colored with $\lfloor \frac{3g-5}{4} \rfloor$ colors so that every color appears on the boundary of each face.*

3.3 Application to the geometric Set k-cover

Unfortunately, it seems we cannot directly use Theorem 3.2 in the geometric Set k-cover problem. Given a subset of hyperedges in the modeling graph H, i.e. a subset of disks/targets D' and a subset of vertices, i.e. sensors, S', the number of incidences between sensors and targets (incidence happens when a sensor monitors a target) can be as bad as quadratic. If all the disks have nonempty intersection and all the sensors belong to this nonempty intersection, then the number of incidences is $|D'| \times |S'|$ (intuitively, this should not present a problem for the solution of the geometric Set k-cover problem since coloring just these sensors will take care of a whole lot of targets).

Note though that the assertion of Theorem 3.2 is stronger than what we need. It asserts that every color

is present at every hyperedge, i.e. every partition of sensors monitors all the targets. As we pointed out, this might not always be possible, e.g. when a target is covered by a single sensor only, the set of sensors cannot be partitioned in this strong sense at all. Yet, if the set of sensors is well spread out so that the precondition of Theorem 3.2 is satisfied, then the theorem guarantees the existence of a partitioning of the sensors in the strong sense. Namely, if a target is covered by at least g sensors, the sensors can be partitioned into $\lfloor \frac{3g-5}{4} \rfloor$ groups so that every target is monitored by a sensor in every group.

4 Conclusions

In this paper we presented two approaches that yield immediate partial results for the geometric Set k-cover problem. The first approach, by Aloupis *et al.* [1], considers the coloring of range spaces asking for the minimum number of colors $c(k)$ so that every finite restriction of the range space has a $c(k)$-coloring so that every range is k-colorful. This relaxation of using more than k colors is meaningful since using exactly k colors might not always be possible. Bounding $c(k)$ for the range space of disks, $\mathcal{D}$, gives close to $\frac{1}{24}$-approximation of the geometric Set k-cover problem.

The second approach, by Alon *et al.* [2], considers coloring of planar graphs all of whose faces have at least g vertices on their boundary. Every such graph can be colored with roughly $\frac{3g}{4}$ colors so that every color appears on every face of the graph. This is stronger than maximizing the total time targets are covered, namely every target is monitored in every time slot. Using Alon *et al.*'s result, if for every subset of sensors S' and targets T' the number of incidences between them is bounded by $i(S', T') \leq 2(|S'| + |T'|)$, and furthermore every target is covered by at least g sensors, we can obtain a cover decomposition into roughly $\frac{3g}{4}$ covers.

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