Computational Methods Homework #2
CMSC / AMSC 460, Fall 2018 Due 11:59PM, October 4, 2018

Problem 1. (15) Let $x_0, x_1, \dots, x_{n-1}, x_n$ define a set of intervals on the real line, where all intervals have the same length, $h = x_j - x_{j-1}$. It has been shown in class that the computation of the coefficients required for piecewise cubic spline interpolation requires the solution of a linear system of equations

$$\begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-3} \\ c_{n-2} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-3} \\ r_{n-2} \end{bmatrix}$$

where $r_i = (3/h)(f[x_i, x_{i+1}] - f[x_{i-1}, x_i])$. The matrix, denoted T, is tridiagonal with $n-2$ rows and columns; in particular, all the entries except those on the diagonal and in the first subdiagonal and superdiagonal are 0. (This equation corresponds to the so-called "free-boundary" condition in which $c_0 = c_{n-1} = 0$.)

a. Show that $T = LL^T$ where L is a triangular matrix with nonzero entries only on the diagonal and first subdiagonal.

b. Describe how to compute the solution vector $\mathbf{c}$ using the factors L and L^T. This description should be accompanied by a pseudocode in MATLAB style that could be used to compute the solution.

Problem 2. (10)
Given a function f defined on $[a, b]$, specify the composite Simpson rule for computing an approximation $Q_{S,C}(f)$ to $I(f) \equiv \int_1^b f(x)\,dx$ and derive an expression for a bound on the error $|I(f) - Q_{S,C}(f)|$.

Problem 3. (20)
a. Consider the following approach for deriving a quadrature rule for approximating an integral $I(f) = \int_0^1 f(x)\,dx$. Let

$$Q(f) = a_0 f(0) + a_1 f(1/4) + a_2 f(1/2) + a_3 f(3/4) + a_4 f(1).$$

We would like $Q(f)$ to be equal to $I(f)$ whenever f is a polynomial of degree at most 4. Use this condition to derive a system of five equations for the five unknowns a_0, a_1, a_2, a_3, a_4.

b. Write a MATLAB program to solve the system and identify the coefficents.

c. How does this quadrature rule relate to those discussed in class? What happens if this rule is applied to a polynomial of degree 5?

Problem 4. (20)
Write a MATLAB program that implements the composite trapezoidal rule to approximate

$$\ln 10 = \int_1^{10} \frac{1}{t}\,dt$$

using n equal sized intervals $[x_{i-1}, x_i]$, $1 \le i \le n$. If $h = 1/n$, demonstrate numerically how the error $e_n \equiv |I(f) - Q(f)|$ behaves as a function of h. This should be done by applying the rule for a sequence of values of n that are doubling in size, say $n = 8, 16, \ldots, 1024$ (so that the interval width h is divided by 2) and showing that the errors are reduced by an appropriate factor with each successive refinement.

To make this demonstration clear, you should print a table showing, for each n, the values of n, h, the computed estimates, the error e_n, and the ratio of errors e_{n-1}/e_n.

Problem 5. (20)

a. The normal distribution function used in statistics is defined to be

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

Use an adaptive quadrature rule of your choice to plot $\mathcal{N}(x)$ for $x \in [-4, 4]$. To do this, you need to replace the lower limit $-\infty$ in the integral with a finite number. For the plot, use the value determined from your experiments in part (b) below.

b. Explore the accuracy of the computed answer as a function of the value chosen for the lower limit. That is, for several choices of the lower limit, identify the maximum error for a set of points in $[-5, 5]$. You can use as an accurate answer the values obtained using -50 as the lower limit and a tolerance of 10^{-14} for the adaptive quadrature.