Advanced Numerical Linear Algebra | Homework 2
AMSC/CMSC 763 | Due October 11, 2019, 5PM

Problem 1.

a. Show that the eigenvalues and eigenvectors of the matrix of order n given by

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

are $\left\{(\lambda_k, v^{(k)}),\right\}$, where for $k = 1, 2 \ldots, n$, $\lambda_k = 4\sin^2\left(\frac{k\pi h}{2}\right)$, $\left[v^{(k)}\right]_j = \sin(k\pi x_j)$, with $h = \frac{1}{n+1}$ and $x_j = \frac{j}{n+1}$, $j = 1, \ldots, n$.

b. Let $D = 2I$, the diagonal matrix taken from the diagonal of A, and let $Q = \frac{1}{\omega}D$ be the damped Jacobi smoothing operator. The eigenvectors can be divided into two sets, one containing "smooth modes" ($k \le \frac{n}{2}$) and the other containing "oscillatory modes" ($\frac{n}{2} \le k \le n$). (For k even, the middle choice here can be viewed as either the most oscillatory smooth mode or the smoothest oscillatory mode.)

The optimal damping parameter ω is the one for which the damping operator $I - Q^{-1}A$ does an equally good job of reducing the error in the most oscillatory eigenvector and the smoothest oscillatory one. Derive this optimal parameter.

Problem 2.

a. Given a linear system of equations $Ax = b$ where A is nonsingular, show that for $k < n$, the kth residual $r_k = b - Ax_k$ produced using the GMRES method satisfies

$$r_k = p_k(A)r_0$$

where $p_k(t)$ is a polynomial of degree k that satisfies $p_k(0) = 1$ and also has the property that

$$\|r_k\|_2 = \min_{p_k \in \Pi_k} \|p_k(A)r_0\|_2$$

where Π_k is the set of all polynomials of degree k that have the value 1 at the origin.

b. If A is diagonalizable, show that

$$\|r_k\|_2 \le c \min_{p_k \in \Pi_k} \max_{\lambda \in \sigma(A)} |p_k(\lambda)| \|r_0\|_2$$

where $\sigma(A)$ is the set of eigenvalues of A and c is independent of k.

c. It may happen that $r_k = 0$ for $k < n$. Describe a way this could occur.

Problem 3. Suppose $A\mathbf{u} = \mathbf{f}$ is a linear system of equations in which the coefficient matrix A is symmetric and positive-definite. Let $Q = GG^T$ be a symmetric positive-definite preconditioner; note that no assumption is made about the structure of G other than that Q admits a factorization of this type. Given this (formal) factorization, the unpreconditioned conjugate gradient algorithm could be applied to

$$G^{-1}AG^{-T}\mathbf{v} = G^{-1}\mathbf{f}, \quad \mathbf{v} = G^T\mathbf{u}.$$

Use this fact to derive the *preconditioned conjugate gradient algorithm* (PCG) given below. This shows that the extra computation required by PCG at each step is a solution of a system with coefficient matrix Q. This may or may not depend on the factorization.

THE PRECONDITIONED CONJUGATE GRADIENT METHOD
Choose $\mathbf{u}^{(0)}$, compute $\mathbf{r}^{(0)} = \mathbf{f} - A\mathbf{u}^{(0)}$, solve $Q\mathbf{z}^{(0)} = \mathbf{r}^{(0)}$, set $\mathbf{p}^{(0)} = \mathbf{z}^{(0)}$
`for` $k = 0$ `until` convergence `do`
  $\alpha_k = \langle \mathbf{z}^{(k)}, \mathbf{r}^{(k)} \rangle / \langle A\mathbf{p}^{(k)}, \mathbf{p}^{(k)} \rangle$
  $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + \alpha_k \mathbf{p}^{(k)}$
  $\mathbf{r}^{(k+1)} = \mathbf{r}^{(k)} - \alpha_k A\mathbf{p}^{(k)}$
  <Test for convergence>
  Solve $Q\mathbf{z}^{(k+1)} = \mathbf{r}^{(k+1)}$
  $\beta_k = \langle \mathbf{z}^{(k+1)}, \mathbf{r}^{(k+1)} \rangle / \langle \mathbf{z}^{(k)}, \mathbf{r}^{(k)} \rangle$
  $\mathbf{p}^{(k+1)} = \mathbf{z}^{(k+1)} + \beta_k \mathbf{p}^{(k)}$
`enddo`

Problem 4. The MATLAB code given below generates a system of equations $Au = f$ corresponding to finite-difference discretization of the Poisson equation

$$-(u_{xx} + u_{yy}) = 1$$

on an L-shaped domain Ω, subject to boundary conditions $u = 0$.

```
G = numgrid('L',n);
A = delsq(G);
h = 1/(n+1);
N = size(A,1);
f = h^2*ones(N,1);
```

The domain can be visualized using the command `spy(G)`.

For the three values of $n = 32$, 64 and 128, do the following:

- Solve the linear system using the unpreconditioned conjugate gradient method. The solver should stop at the first step k for which the residual norm $\|r_k\|_2$ satisfies $\|r_k\|_2/\|f\|_2 \leq 10^{-6}$.

- Solve it using the preconditioned conjugate gradient method with preconditioner D, the diagonal matrix containing the diagonal entries of A.
- Solve it using the preconditioned conjugate gradient method using the incomplete Cholesky preconditioner as defined by the routine `ichol` in MATLAB.
- On one graph, plot the relative residual norms $\{\|r_k\|_2/\|r_0\|_2\}$ obtained for all three problems using the unpreconditioned method, and on a separate graph, plot the relative residual norms obtained using the incomplete Cholesky preconditioner.
- Explain what happens with the diagonal preconditioner.

Comment: You may use any software you want to do this problem as long as you document what you use. It is not necessary to use MATLAB although if you prefer another language, it might still be easier to generate the problem using MATLAB.