Advanced Numerical Linear Algebra Homework 5
AMSC/CMSC 763 Due December 11, 2019
5:00 PM

Problem 1.

a. Let $X = U\Sigma V^T$ where $U \in \mathbb{R}^{m\times\ell}$ and $V \in \mathbb{R}^{\ell\times n}$ each have orthogonal columns, $\Sigma = diag\,[\sigma_1, \ldots \sigma_\ell]$ with $\sigma_1 \geq \cdots \geq \sigma_\ell \geq 0$, and $\ell \leq \min(m, n)$. Show that $\|X\|_2 = \sigma_1$. Use this result to show that $\|A\|_2 = \|A^T\|_2$ when A is a rectangular matrix.

b. For symmetric matrices M and N, the notation $M \preceq N$ means that $N - M$ is positive semi-definite. Show that if M and N are both positive semidefinite, then $M \preceq N$ implies that $\|M\|_2 \leq \|N\|_2$.

Problem 2. An orthogonal projector on $\mathbb{R}^n$ is a symmetric matrix P that satisfies $P^2 = P$.

a. Show that if P is an orthogonal projector, then $0 \preceq P \preceq I$.

b. Show that all the singular values of P are either 1 or 0.

c. Show that P is uniquely determined by its range.
Hint: If Q is another orthogonal projector with the same range, then use the singular value decompositions of P and Q to show that $P = Q$.

Problem 3. Implement the following randomized SVD algorithm and explore how it works for the matrix in the file `hw5.mat`. Test it for $q = 0$, 1 and 2. In particular, compare the singular values obtained by this method to those obtained from the true SVD, and examine the (norm of the) difference between the true left singular matrix U and the one obtained by this method.
The algorithm comes from p. 227 of Halko, et al., SIAM Review 53, pp. 217-288, 2011.
Stage A:
1. Generate an $n \times 2k$ Gaussian test matrix Ω.
2. Form $Y = (AA^T)^q A\Omega$ by multiplying alternatively with A and A^T.
3. Construct a matrix Q whose columns form an orthonormal basis for the range of Y.
Stage B:
4. Form $B = Q^T A$.
5. Compute an SVD of the small matrix, $B = \widetilde{U}\Sigma V^T$
6. Set $U = Q\widetilde{U}$.