1 $f(55,34) \le \frac{151}{374}$

Theorem 1.1 $f(55,34) \le \frac{151}{374}$.

Proof:

Assume there is an $(55,34)$-procedure. We show that there is a piece $\le \frac{151}{374}$.

We note that $\frac{55}{34} = \frac{605}{374}$.

Case 1: Some student gets ≥ 5 shares. $sh_S(5) \le \frac{55}{34\times 5} < \frac{151}{374}$.

Case 2: Some student gets ≤ 2 shares. $B(sh_L(2)) \le 1 - \frac{55}{34\times 2} < \frac{151}{374}$

Case 3: Every muffin is cut in 2 pieces and every student gets either 3 or 4 shares. The total number of shares is 110. Let s_3 (s_4) be the number of 3-students (4-students).

$$\begin{aligned} 3s_3 + 4s_4 = &\ 110 \\ s_3 + s_4 = &\ 34 \end{aligned}$$

So $s_3 = 26$ and $s_4 = 8$.

Case 3.1: There is a 3-share $x \le \frac{159}{374}$. $M_L(x) \ge \frac{\frac{605}{374} - \frac{159}{374}}{2} = \frac{223}{374}$ so $B(M_L(x)) \le 1 - \frac{223}{374} = \frac{151}{374}$

Case 3.2: There is a 4-share $x \ge \frac{152}{374}$. $M_S(x) \le \frac{\frac{605}{374} - \frac{152}{374}}{3} = \frac{151}{374}$.

Case 3.3: There is a 3-share $x \ge \frac{223}{374}$. $B(x) \le 1 - \frac{223}{374} = \frac{151}{374}$

Case 3.4: There is a 4-share $x \le \frac{151}{374}$. This one is self-explanatory.

Case 3.5: All 3-shares are in $(\frac{159}{374}, \frac{223}{374})$ and all 4-shares are in $(\frac{151}{374}, \frac{152}{374})$.

The following picture captures what we know so far.

(	$---$	)[	$---$	](	$---$	)
$\frac{151}{374}$	4-shs	$\frac{152}{374}$	No shs	$\frac{159}{374}$	3-shs	$\frac{223}{374}$

Since there are no shares in $[\frac{152}{374}, \frac{159}{374}]$, there are no shares in $B([\frac{152}{374}, \frac{159}{374}]) = [\frac{215}{374}, \frac{222}{374}]$

The following picture captures what we know so far.

(	$---$	)[	$---$	](	$---$	)[	$---$	](	$---$	)
$\frac{151}{374}$	4-shs	$\frac{152}{374}$	No shs	$\frac{159}{374}$	S3-shs	$\frac{215}{374}$	No shs	$\frac{222}{374}$	L3-shs	$\frac{223}{374}$

There are $4s_4 = 32$ 4-shares so there are 32 L3-shares (B is a bijection between 4-shares and L3-shares). Since there are 110 shares total that leaves 46 S3 shares. The interval of the S3 shares is symmetric since its midpoint is $\frac{1}{2}\frac{374}{374}$. Hence $(\frac{159}{374}, \frac{187}{374})$ and $(\frac{187}{374}, \frac{215}{374})$ have the same number of shares. It might not be 23 since there could be shares of size $\frac{187}{374}$.

Def 1.2

1. $I_1 = (\frac{159}{374}, \frac{187}{374})$
2. $I_2 = (\frac{187}{374}, \frac{215}{374})$ ($|I_1| = |I_2|$)
3. $I_3 = (\frac{222}{374}, \frac{223}{374})$ ($|I_3| = 32$)

Claim 1:

1. The following are the only students who are allowed.
   (a) $e(1, 2, 2)$
   (b) $e(1, 2, 3)$
   (c) $e(1, 3, 3)$
   (d) $e(2, 2, 2)$
   (e) $e(2, 2, 3)$
2. There are no shares in $[\frac{161}{374}, \frac{167}{374}]$
3. There are no shares in $[\frac{207}{374}, \frac{213}{374}]$ (this follows from the prior part and buddying).

Proof of Claim 1:

1) We establish that some students are impossible.

A $e(1,1,3)$-student has less than $2 \times \frac{187}{374} + \frac{223}{374} = \frac{597}{374}$

A $e(2,3,3)$-student has more than $\frac{187}{374} + 2 \times \frac{222}{374} = \frac{631}{374}$

The result follows.

2) We look at which I_1-shares are used

A $e(1,2,2)$ student uses I_1-share $> \frac{605}{374} - 2 \times \frac{215}{374} = \frac{175}{374}$

A $e(1,2,3)$ student uses I_1-shares $> \frac{605}{374} - \times \frac{215}{374} - \frac{223}{374} = \frac{167}{374}$

A $e(1,3,3)$ student uses I_1-shares $< \frac{605}{374} - 2 \times \frac{222}{374} = \frac{161}{374}$

The result follows.

End of Proof of Claim 1

We redefine the intervals.

Def 1.3

1. $I_1 = (\frac{159}{374}, \frac{161}{374})$
2. $I_2 = (\frac{167}{374}, \frac{187}{374})$
3. $I_3 = (\frac{187}{374}, \frac{207}{374})$ $(|I_2| = |I_3)$
4. $I_4 = (\frac{213}{374}, \frac{215}{374})$ $(|I_1| = |I_4|)$
5. $I_5 = (\frac{222}{374}, \frac{223}{374})$ $(|I_5| = 32)$

Claim 2:

1. The following are the only students who are allowed.
   (a) $e(1,5,5)$
   (b) $e(2,3,5)$

(c) $e(2,4,5)$

(d) $e(3,3,3)$

(e) $e(3,3,4)$

(f) $e(3,3,5)$

2. There are no shares in $[\frac{170}{374}, \frac{175}{374}]$

3. There are no shares in $[\frac{199}{374}, \frac{204}{374}]$ (this follows from the prior part and buddying).

Proof of Claim 2: 1) We establish that some students are impossible.

A $e(1,4,5)$-student has less than $\frac{161}{374} + \frac{215}{374} + \frac{223}{374} = \frac{599}{374}$

A $e(2,2,5)$-student has less than $2 \times \frac{187}{374} + \frac{223}{374} = \frac{597}{374}$

A $e(2,5,5)$-student has more than $\frac{167}{374} + 2 \times \frac{222}{374} = \frac{611}{374}$

A $e(3,4,4)$-student has more than $\frac{187}{374} + 2 \times \frac{213}{374} = \frac{613}{374}$

A $e(2,4,4)$-student has less than $\frac{187}{374} + 2 \times \frac{215}{374} = \frac{617}{374}$

The result follows.

2) We look at which I_2-shares are used

A $e(2,3,5)$-student uses I_2-share $> \frac{605}{374} - \frac{207}{374} - \frac{223}{374} = \frac{175}{374}$

A $e(2,4,5)$-student uses I_2-share $< \frac{605}{374} - \frac{213}{374} - \frac{222}{374} = \frac{170}{374}$

The result follows.

End of Proof of Claim 2

We redefine the intervals.

Def 1.4

1. $I_1 = (\frac{159}{374}, \frac{161}{374})$

2. $I_2 = (\frac{167}{374}, \frac{170}{374})$

3. $I_3 = (\frac{175}{374}, \frac{187}{374})$

4. $I_4 = (\frac{187}{374}, \frac{199}{374})$ ($|I_3| = |I_4|$)

5. $I_5 = (\frac{204}{374}, \frac{207}{374})$ ($|I_2| = |I_5|$)

6. $I_6 = (\frac{213}{374}, \frac{215}{374})$ ($|I_1| = |I_6|$)

7. $I_7 = (\frac{222}{374}, \frac{223}{374})$ ($|I_7| = 32$)

Claim 3:

1. The following are the only students who are allowed.

   (a) $e(1, 7, 7)$

   (b) $e(2, 6, 7)$

   (c) $e(3, 4, 7)$

   (d) $e(3, 5, 6)$

   (e) $e(3, 5, 7)$

   (f) $e(3, 6, 6)$

   (g) $e(4, 4, 6)$

   (h) $e(4, 4, 7)$

   (i) $e(4, 5, 5)$

   (j) $e(4, 5, 6)$

2. There are no shares in $[\frac{179}{374}, \frac{183}{374}]$

3. There are no shares in $[\frac{191}{374}, \frac{195}{374}]$ (this follows from the prior part and buddying).

Proof of Claim 3:

1) We establish that some students are impossible.

A $e(1,6,7)$-student has less than $\frac{161}{374}+\frac{215}{374}+\frac{223}{374}=\frac{599}{374}$

A $e(2,5,7)$-student has less than $\frac{170}{374}+\frac{207}{374}+\frac{223}{374}=\frac{600}{374}$

A $e(2,6,6)$-student has less than $\frac{170}{374}+2\times\frac{215}{374}=\frac{600}{374}$

A $e(2,7,7)$-student has more than $\frac{167}{374}+2\times\frac{222}{374}=\frac{611}{374}$

A $e(3,3,7)$-student has less than $2\times\frac{187}{374}+\frac{223}{374}=\frac{597}{374}$

A $e(3,4,6)$-student has more than $\frac{187}{374}+\frac{199}{374}+\frac{215}{374}=\frac{601}{374}$

A $e(3,5,5)$-student has less than $\frac{187}{374}+2\times\frac{207}{374}=\frac{601}{374}$

A $e(3,6,7)$-student has more than $\frac{175}{374}+\frac{213}{374}+\frac{222}{374}=\frac{610}{374}$

A $e(4,4,5)$-student has less than $2\times\frac{199}{374}+\frac{207}{374}=\frac{605}{374}$. WOW!

A $e(4,5,7)$-student has more than $\frac{187}{374}+\frac{204}{374}+\frac{222}{374}=\frac{613}{374}$

A $e(4,6,6)$-student has more than $\frac{187}{374}+2\times\frac{213}{374}=\frac{613}{374}$

A $e(5,5,5)$-student has more than $3\times\frac{204}{374}=\frac{612}{374}$

The result follows

2) We look at which I_3-shares are used

A $e(3,4,7)$-student has I_3-share $>\frac{605}{374}-\frac{199}{374}-\frac{223}{374}=\frac{183}{374}$

A $e(3,5,6)$-student has I_3-share $>\frac{605}{374}-\frac{207}{374}-\frac{215}{374}=\frac{183}{374}$

A $e(3,5,7)$-student has I_3-share $<\frac{605}{374}-\frac{204}{374}-\frac{222}{374}=\frac{179}{374}$

A $e(3,6,6)$-student has I_3-share $<\frac{605}{374}-2\times\frac{213}{374}=\frac{179}{374}$

The result follows.

End of Proof of Claim 3

We redefine the intervals.

Def 1.5

1. $I_1=(\frac{159}{374},\frac{161}{374})$

2. $I_2=(\frac{167}{374},\frac{170}{374})$

3. $I_3 = (\frac{175}{374}, \frac{179}{374})$

4. $I_4 = (\frac{183}{374}, \frac{187}{374})$

5. $I_5 = (\frac{187}{374}, \frac{191}{374})$ $(|I_4| = |I_5|)$

6. $I_6 = (\frac{195}{374}, \frac{199}{374})$ $(|I_3| = |I_6|)$

7. $I_7 = (\frac{204}{374}, \frac{207}{374})$ $(|I_2| = |I_7|)$

8. $I_8 = (\frac{213}{374}, \frac{215}{374})$ $(|I_1| = |I_8|)$

9. $I_9 = (\frac{222}{374}, \frac{223}{374})$ $(|I_9| = 32)$

Claim: The following are the only students who are allowed.

1. $e(1, 9, 9)$

2. $e(2, 8, 9)$

3. $e(3, 7, 9)$

4. $e(3, 8, 8)$

5. $e(4, 7, 8)$

6. $e(5, 7, 8)$

7. $e(6, 6, 8)$

8. $e(6, 7, 7)$

We establish that some students are impossible.

A $e(1, 8, 9)$-student has less than $\frac{161}{374} + \frac{215}{374} + \frac{223}{374} = \frac{599}{374}$.

A $e(2, 8, 8)$-student has less than $\frac{170}{374} + 2 \times \frac{215}{374} = \frac{600}{374}$.

A $e(2,9,9)$-student has more than $\frac{167}{374} + 2 \times \frac{222}{374} = \frac{611}{374}$

A $e(3,7,8)$-student has less than $\frac{179}{374} + \frac{207}{374} + \frac{215}{374} = \frac{601}{374}$

A $e(3,8,9)$-student has more than $\frac{175}{374} + \frac{213}{374} + \frac{222}{374} = \frac{610}{374}$

A $e(2,7,9)$-student has less than $\frac{170}{374} + \frac{207}{374} + \frac{223}{374} = \frac{600}{374}$

A $e(4,7,9)$-student has more than $\frac{183}{374} + \frac{204}{374} + \frac{222}{374} = \frac{609}{374}$

A $e(4,8,8)$-student has more than $\frac{183}{374} + 2 \times \frac{213}{374} = \frac{609}{374}$

A $e(5,7,7)$-student has more than $\frac{191}{374} + 2 \times \frac{207}{374} = \frac{605}{374}$. WOW!

A $e(3,6,9)$-student has less than $\frac{179}{374} + \frac{199}{374}\frac{223}{374} = \frac{601}{374}$

A $e(6,6,7)$-student has less than $2 \times \frac{199}{374} + \frac{207}{374} = \frac{605}{374}$. WOW!

A $e(6,6,9)$-student has more than $2 \times \frac{195}{374} + \frac{222}{374} = \frac{612}{374}$

A $e(6,7,8)$-student has more than $\frac{195}{374} + \frac{204}{374} + \frac{213}{374} = \frac{612}{374}$

A $e(7,7,7)$-student has more than $3 \times \frac{204}{374} = \frac{612}{374}$

The result follows.

End of Proof of Claim

Let:

1. $a = |e(1,9,9)|$
2. $b = |e(2,8,9)|$
3. $c = |e(3,7,9)|$
4. $d = |e(3,8,8)|$
5. $e = |e(4,7,8)|$
6. $f = |e(5,7,8)|$
7. $g = |e(6,6,8)|$
8. $h = |e(6,7,7)|$

Since $|I_4| = |I_5|$, $e = f$

Since $|I_1| = |I_8|$, $a = b + 2d + e + f + g = b + 2d + 2f + g$.

Since $|I_2| = |I_7|$, $b = c + f + 2h = c + 2f + 2h$.

Combining the last two equations you get

$$a = (c + 2f + 2h) + 2d + 2f + g = c + 2d + 4f + g + 2h$$

Since $|I_3| = |I_6|$, $c + d = 2g + h$ which we rewrite as $c = 2g + h - d$.

Combining this with the expression for a we get

$$a = c+2d+e+4f+g+2h = d+(c+d)+e+4f+g+2h = 2g+h+e+4f+g+2h = e+4f+3g+3h$$

Since $|I_9| = 32$, $2a + b + c = 32$

Since $s_3 = 26$, $a + b + c + d + 2f + g + h = 26$

Subtract these two to get

$$a - d - 2f - g - h = 6$$

$$a = 6 + d + 2f + g + h$$

$$a = c + 2d + 4f + g + 2h$$

SO

$$6 + d + 2f + g + h = c + 2d + 4f + g + 2h$$

$$6 + d + 2f + h = c + 2d + 4f + 2h$$

$$6 + 2f + h = c + d + 4f + 2h$$

$$6 + h = c + d + 2f + 2h$$

$$6 = c + d + 2f + h$$

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We express the system in the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & -1 & -1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 26 \\ 32 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By adding positive integer multiples of the first 5 columns in order to reduce the bottom 4 rows to 0, we see that this is equivalent to solving

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 4 & 8 & 8 & 8 \\ 2 & 3 & 4 & 4 & 5 & 10 & 10 & 10 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 26 \\ 32 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which clearly has no solution since $\frac{26}{32} \neq \frac{8}{10}$. ∎