If $|A| \subseteq \{1,\ldots,N\}$ is Large Enough then A has a 3-AP

A Summary of Results

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1 Introduction

Notation 1.1 $[N]$ is the set $\{1,\ldots,N\}$.

There has been some results along the following lines:

Theorem 1.2 *Let $k \in \mathbb{N}$. Let $N \in \mathbb{N}$ be large. Let $A \subseteq [N]$ and $|A| \ge r_k(N)$ then A has a k-AP.*

In Section 2 we present a list of such results for $k = 3$. We tried to be complete; however, if you know of a result we missed, please let us know. In Section 3 we show that if there exists $b > 1$ such that $r_k(N) \le \frac{N}{(\log N)^b}$ then

$$\sum_{x \in A} \frac{1}{x} \implies A \text{ has a } k\text{-AP}.$$

(One of the Erdős-Turan (ET) conjectures is the implication.) The proof we present is surely well known; however, we have not seen it written down, hence we thought it wise to write it down. We then note that the recent result of Bloom and Sisack solves the ET conjecture for $k = 3$ (this was the motivation for their work). In Section 4 we present the results for r_4 and for r_k. Again we tried to be complete; however, if you know of a result we missed, please let us know.

For reasons of space we list the known results on the next page. Some of the website entries go off the page. This does not matter since you can click on the website and get there. (The Joys of not having to go through a refereeing process!)

2 List of Results on r_3

When we list $r(N)$ we really mean $O(r(N))$. The results all use $\log$. The base does not matter since they are all O-of results.

$r_3(N)$	Author	Paper
$N \times \frac{1}{\log \log N}$	Roth	http://www.cs.umd.edu/~gasarch/TOPICS/vdw/roth.pdf
Notes	Alex Iosevich	http://www.cs.umd.edu/~gasarch/TOPICS/vdw/notes-roth
Notes	Terry Tao	https://terrytao.wordpress.com/2010/04/08/254b-notes
$N \times \frac{1}{(\log N)^{\delta}}$ Fixed small δ :-(	Szemeredi and Health-Brown	http://www.cs.umd.edu/~gasarch/TOPICS/vdw/szlog.pdf http://www.cs.umd.edu/~gasarch/TOPICS/vdw/heathbrown
$N \times \left(\frac{(\log \log N)^{1/2}}{(\log N)^{1/2}}\right)$	Bourgain	http://www.cs.umd.edu/~gasarch/TOPICS/vdw/Bourgainor
$N \times \left(\frac{(\log \log N)^{2}}{(\log N)^{2/3}}\right)$	Bourgain	http://www.cs.umd.edu/~gasarch/BLOGPAPERS/bourgaintw
$N \times \frac{1}{(\log N)^{3/4-o(1)}}$	Sanders	https://arxiv.org/abs/1007.5444
$N \times \left(\frac{(\log \log N)^{6}}{\log N}\right)$ Paper says 5, but proof only works for 6.	Sanders	https://arxiv.org/abs/1011.0104
$N \times \left(\frac{(\log \log N)^{4}}{\log N}\right)$	Bloom	https://arxiv.org/abs/1405.5800
$N \times \left(\frac{(\log \log N)^{3+o(1)}}{\log N}\right)$	Schoen	https://arxiv.org/abs/2005.01145
$N \times \frac{1}{(\log N)^{1-o(1)}}$ Diff Proof than last 3 results	Bloom and Sisask	https://arxiv.org/pdf/1810.12791.pdf
$N \times \frac{1}{(\log N)^{1+\delta}}$ Fixed small δ :-(	Bloom and Sisask	https://arxiv.org/abs/2007.03528

3 Why the New Result Solves the Erdős-Turan Conjecture for $k = 3$

Erdős and Turan together made two conjectures along the lines of

if a set A is large enough then, for all k, A contains an arithmetic progression of length k (a k-AP).

We state both conjectures along with their status.

Conjecture 3.1 *The Erdős-Turan conjectures are as follows:*

1. *Let $\delta > 0$ and $k \in \mathbb{N}$. There exists N_0 such that, for all $N \geq N_0$, for all $A \subseteq [N]$ with $|A| \geq \delta N$, A has a k-AP. This was proven by Szemeredi in 1974. There are now several proofs.*

2. *Let $A \subseteq \mathbb{N}$, $A = \{a_1 < a_2 < \cdots\}$. If $\sum_{n \in \mathbb{N}} \frac{1}{a_n}$ diverges then, for all k, A has a k-AP. This is still open. The recent result on $r(N)$ states above, by Bloom and Sisask, implies that the $k = 3$ case of the conjecture is true. While this implication is well known, we give a proof in this section.*

Notation 3.2 Let $r : \mathbb{N} \to \mathbb{R}$ be monotone increasing. Let $x \in \mathbb{R}$. Then $r^{-1}(x)$ is the unique $y \in \mathbb{N}$ such that

- $r(y) \leq x$
- $r(y+1) > x$.

The following lemma is proven by algebraic manipulations that we omit.

Lemma 3.3 *Let $b > 0$. Let $r(N)$ be such that $r(N) \leq O\big(\frac{N}{(\log N)^b}\big)$. Then, for all $n \in \mathbb{N}$ the following hold:*

1. $n(\log n)^b \leq O(r^{-1}(n))$.

2. $\frac{1}{r^{-1}(n)} \leq O\big(\frac{1}{n(\log n)^b}\big)$ *(this follows from the prior item).*

Lemma 3.4 *Let $r : \mathbb{N} \to \mathbb{R}$ be monotone increasing. Let $A \subseteq \mathbb{N}$ be such that, for all $N \in \mathbb{N}$, $|A \cap [N]| \le r(N)$.*

Let $A = \{a_1 < a_2 < \cdots\}$.

Then, for all $n \in \mathbb{N}$, the following hold:

1. *$n \le r(a_n)$.*
2. *$r^{-1}(n) \le a_n$ (this follows from the prior item).*
3. *$\frac{1}{a_n} \le O\big(\frac{1}{r^{-1}(n)}\big)$ (this follows from the prior item).*
4. *If $r(N) \le \frac{N}{(\log N)^b}$ then $\frac{1}{a_n} \le \frac{1}{n(\log n)^b}$ (this follows from the prior item and Lemma 3.3).*

Proof: For all n, $n = |A \cap [a_n]| \le r(a_n)$. ∎

Lemma 3.5 *Let $k \in \mathbb{N}$. Let $b > 1$ and let $r(N) = O\big(\frac{N}{(\log N)^b}\big)$. Let $A \subseteq \mathbb{N}$ be such that, for all $N \in \mathbb{N}$, $|A \cap [N]| \le r_k(N)$. Let $A = \{a_1 < a_2 < \cdots\}$. Then $\sum_{n\in\mathbb{N}} \frac{1}{a_n}$ converges.*

Proof: Let $A = \{a_1, a_2, \ldots, \}$. Let N_0 and α be such that, for all $N \ge N_0$, $|A \cap [N]| \le O\big(\frac{\alpha N}{(\log N)^b}\big)$. Then by Lemma 3.4 there exists N_0, β such that, for all $N \ge N_0$, $\frac{1}{a_N} \le \frac{\beta}{N(\ln N)^b}$.

We only look at the summation past N_0 which suffices for issues of convergence.

$$\sum_{i=N_0}^{\infty} \frac{1}{a_i} \le \sum_{i=N_0}^{\infty} \frac{\beta}{i(\ln i)^b} \sim \int_{N_0}^{\infty} \frac{\beta dx}{x(\ln x)^b} = \int_{\ln(N_0)}^{\infty} \frac{\beta du}{u^b} \text{ which converges since } b > 1$$

∎

Theorem 3.6 *Let $k \in \mathbb{N}$. Assume there exists $\delta > 0$ such that $r_k(N) \le O\big(\frac{N}{(\log N)^{1+\delta}}\big)$. Let $A \subseteq \mathbb{N}$, $A = \{a_1 < a_2 < \cdots\}$. If $\sum_{n\in\mathbb{N}} \frac{1}{a_n}$ diverges then A contains a k-AP.*

Proof: We prove the contrapositive. Assume A has no k-AP. Then, by our hypothesis, $|A \cap [N]| \le O\left(\frac{N}{(\log N)^{1+\delta}}\right)$. By Lemma 3.5 $\sum_{n\in\mathbb{N}} \frac{1}{a_n}$ converges. ▮

Corollary 3.7 *Let $A \subseteq \mathbb{N}$, $A = \{a_1 < a_2 < \cdots\}$. If $\sum_{n\in\mathbb{N}} \frac{1}{a_n}$ converges then A has a 3-AP.*

Proof: This follows from Theorem 3.6 and the recent result of Bloom and Sisack on the table in the last section. ▮

4 $r_4(N)$ and $r_k(N)$

We state the results on r_4 in the table below. These results do not suffice to show the ET conjecture for $k = 4$.

$r_4(N)$	Author	Paper
$N \times \frac{1}{(\log\log N)^{\delta}}$ Fixed small δ :-(	Gowers	`http://link.springer.com/content/pdf/10.1007/s000390050065.p`
$N \times \frac{1}{e^{\delta\sqrt{\log\log N}}}$ Fixed small δ :-(	Green and Tao	`https://arxiv.org/abs/math/0610604`
$N \times \frac{1}{(\log N)^{\delta}}$	Green and Tao	`https://arxiv.org/abs/1705.01703`

There is only one result for $k = 5$. It falls out of a general result of Gowers that we state in the table below. The table only has one entry; however, in the future we hope to add to it.

$r_k(N)$	Author	Paper
$N \times \frac{1}{(\log\log N)^{\delta_k}}$ Fixed small δ_k :-(	Gowers	`https://www.cs.umd.edu/users/gasarch/TOPICS/vdw/sz-thm-gowers-pr`