Explict Polynomials for $\cos(a\pi/b)$

1 Introduction

In high school we are taught the following:
Table of Some Cosines

1. $\cos(\pi/1) = -1$
2. $\cos(\pi/2) = 0$
3. $\cos(\pi/3) = \frac{1}{2}$
4. $\cos(\pi/4) = \frac{\sqrt{2}}{2}$.
5. $\cos(\pi/6) = \frac{\sqrt{3}}{2}$.

Note that $\cos(\pi/5)$ is missing. In Harold Boas's paper [1] he shows that

$$\cos(\pi/5) = \frac{1+\sqrt{5}}{4}$$

which is twice the golden ratio. More to the point,

it is a reasonable number!

By contrast he later shows that

$$\cos(\pi/7) = \frac{1}{6}\left(1 + \left(\frac{7}{2}\left(-1+3i\sqrt{3}\right)\right)^{1/3} + \left(\frac{7}{2}\left(-1-3i\sqrt{3}\right)\right)^{1/3}\right)$$

which is an unreasonable number. He later goes to show that no other fractional cosine is reasonable. (I am not going to rigorously define *reasonable*.)

In this note we use the techniques in his paper to derive, for all $a, b \in \mathsf{N}$ with $b \neq 0$, an explicit polynomial $p(x) \in \mathsf{Z}[x]$ such that $p(\cos(\frac{a\pi}{b})) = 0$. This establishes that all such numbers are algebraic, which is already known. We will use using Chebyshev polynomials and elementary facts from trigonometry. We list those elementary facts here:

1. For all $k \in \mathsf{N}$, $\cos(\theta) = \cos(\theta + 2k\pi)$.
2. For all $k \in \mathsf{N}$, $\cos(\theta) = -\cos(\theta + (2k+1)\pi)$.
3. $\cos(\theta) = \cos(-\theta)$.
4. $\cos(0) = 1$.
5. $\cos(\pi) = -1$.
6. For all $0 \leq \theta \leq \pi/2$, $\cos(\theta) \geq 0$.

7. For all $\pi/2 \le \theta \le \pi$, $\cos(\theta) \le 0$.
8. For all $\pi \le \theta \le 3\pi/2$, $\cos(\theta) \le 0$.
9. For all $3\pi/2 \le \theta \le 2\pi$, $\cos(\theta) \ge 0$.

We will *not* derive the values of cosine in the order in the table. We will start with what we want to use and see what pops out.

2 Chebyhev Polynomaials of the First Kind

Def 2.1 The Chebyshev Polynomial of the first kind is, for all $n \in \mathsf{N}$,

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2-1)^k x^{n-2k} x.$$

The following are well known.

Theorem 2.2

1. *For all $n \in \mathsf{N}$, $T_n(\cos(\theta)) = \cos(n\theta)$.*
2. *(a)* $T_2(x) = 2x^2 - 1$
   (b) $T_3(x) = 4x^3 - 3x$
   (c) $T_4(x) = 8x^4 - 8x^2$
   (d) $T_5(x) = 16x^5 - 20x^3 + 5x$
   (e) $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
   (f) $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$
   (g) $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2$
   (h) $T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$

3 When is $\cos\left(\frac{a\pi}{b}\right) = \cos\left(\frac{na\pi}{b}\right)$?

Lemma 3.1

1. *Let $n \in \mathsf{N}$. For all*

$$\theta \in \left\{\frac{2k\pi}{n-1} : k \in \mathsf{N}\right\} \bigcup \left\{\frac{2k\pi}{n+1} : k \in \mathsf{N}\right\}$$

$\cos(\theta) = \cos(n\theta)$. *(If $n = 1$ then just take the second unionand.)*

2. *The n roots of $T_n(x) - x = 0$ are*

$$\left\{\cos\left(\frac{2k\pi}{n-1}\right) : 1 \le k \le \frac{n-1}{2}\right\} \bigcup \left\{\cos\left(\frac{2k\pi}{n+1}\right) : 0 \le k \le \frac{n}{2}\right\}.$$

(If $n = 1$ then just take the second unionand.)

Proof:
1)

For the first unionand notice that

$$\cos\left(\frac{2k\pi}{n-1}\right) = \cos\left(\frac{2k\pi}{n-1} + 2k\pi\right) = \cos\left(\frac{n2k\pi}{n-1}\right).$$

For the second unionand notice that

$$\cos\left(\frac{2k\pi}{n+1}\right) = \cos\left(-\frac{2k\pi}{n+1}\right) = \cos\left(2\pi k - \frac{2k\pi}{n+1}\right) = \cos\left(\frac{n2k\pi}{n+1}\right).$$

2) By Theorem 2.2 and Part 1 we have that all of the numbers specified are roots of $T(x) - x = 0$. Simple algebra shows that all of the angles are distinct, so all of the roots are distinct. Since all of the numbes are cosines of angles in $[0, \pi]$, and cosine is injective in that interval, we have exactly n roots. ∎

4 Using $\cos(2\theta)$ to get $\cos(\pi/3), \cos(2\pi/3)$

By Lemma 3.1.2 the 2 roots of $T_2(x) - x = 0$ are

$$\{\cos(0), \cos(2\pi/3)\}.$$

Since $T_2(x) = x = 2x^2 - x - 1 = (x-1)(2x+1)$ we have

$$\{1, -1/2\} = \{\cos(0), \cos(2\pi/3)\}.$$

Since $\cos(0) = 1$, we have

1. $\cos(2\pi/3) = -\frac{1}{2}$. Root of $2x + 1$.
2. $\cos(\pi/3) = -\cos(\pi - \pi/3) = -\cos(2\pi/3) = \frac{1}{2}$. $-2x + 1$.

5 Using $\cos(3\theta)$ to get $\cos(\pi/2)$

By Lemma 3.1.2 the 3 roots of $T_3(x) - x = 0$ are

$$\{\cos(0), \cos(\pi/2), \cos(\pi)\}.$$

Since $T_3(x) - x = 4x^2 - 4x = 4x(x-1)(x+1)$

$$\{0, 1, -1\} = \{\cos(0), \cos(\pi/2), \cos(\pi)\}.$$

We know $\cos(0) = 1$ and $\cos(\pi) = -1$, so

1. $\cos(\pi/2) = 1$. Root of $x - 1 = 0$.

6 Using $\cos(4\theta)$ to get $\cos(a\pi/5)$

By Lemma 3.1.2 the 4 roots of $T_4(x) - x = 0$ are

$$\left(\forall\theta \in \left\{0, \frac{2\pi}{5}, \frac{4\pi}{5}\right\}\right)\left[\cos(\theta) = \cos(4\theta)\right].$$

By the definition of T_4 we know that if

$$\left(\forall\theta \in \left\{0, \frac{2\pi}{5}, \frac{4\pi}{5}\right\}\right)\left[\theta \text{ is a root of } x = T_4(x)\ \right]$$

$$x = 8x^4 - 8x^2 + 1$$

$$(2x+1)(x-1)(4x^2 + 2x - 1) = 0$$

The roots are $-\frac{1}{2}, 1, \frac{-1+\sqrt{5}}{4}, \frac{-1-\sqrt{5}}{4}$
Since $\cos(0) = 1$ and $\cos(2\pi/3) = -\frac{1}{2}$ we have that

$$\left\{\cos\left(\frac{2\pi}{5}\right), \cos\left(\frac{4\pi}{5}\right), \right\} = \left\{\frac{-1+\sqrt{5}}{4}, \frac{-1-\sqrt{5}}{4}\right\}.$$

Since $\cos(2\pi/5) > 0$ and $\cos(4\pi/5) < 0$ we have $\cos(2\pi/5) = \frac{-1+\sqrt{5}}{4}$ and $\cos(4\pi/5) = \frac{-1-\sqrt{5}}{4}$. We now give all cosines of all fractions with denomiator 5.

Hence

$$\cos(\pi/5) = \cos(-\pi/5) = -\cos(\pi - \pi/5) = -\cos(4\pi/5) = \frac{-1-\sqrt{5}}{4} = \frac{1+\sqrt{5}}{4}.$$

$$\cos(3\pi/5) = \cos(-3\pi/5) = -\cos(\pi - 3\pi/5) = -\cos 2\pi/5 = \frac{1-\sqrt{5}}{4}.$$

In summary and in order

1. $\cos(\pi/5) = \frac{1+\sqrt{5}}{4}$. Root of $4x^2 - 2x - 1 = 0$.

2. $\cos(2\pi/5) = \frac{-1+\sqrt{5}}{4}$. Root of $4x^2 + 2x - 1 = 0$.

3. $\cos(3\pi/5) = \frac{1-\sqrt{5}}{4}$. Root of $4x^2 - 2x - 1 = 0$.

4. $\cos(4\pi/5) = \frac{-1-\sqrt{5}}{4}$. Root of $4x^2 + 2x - 1 = 0$.

7 For $a, b \in \mathsf{N}$, $\cos(a\pi/b)$ is Algebraic

The above examples show that all of the number $\cos(\pi/2)$, $\cos(\pi/3)$, $\cos(\pi/4)$, $\cos(\pi/5)$, and $\cos(\pi/6)$ are algebraic. Are all numbers of the form $\cos(\pi/b)$ algebraic? If so then easily so are all numbers of the form $\cos(a\pi/b)$. The answer is Yes.

Theorem 7.1

1. *For all $n \in \mathsf{N}$, $\cos(\pi/n)$ is algebraic.*

2. *For all $n, m \in \mathsf{N}$, $\cos(m\pi/n)$ is algebraic. (This follows from part 1 and the cosine addition formula.*

3. *For all $n, m \in \mathsf{N}$, $\sin(m\pi/n)$ is algebraic. (This follows from part 2 and the relationship between sine and cosine, and the closure of the algebraic numbers under square roots.)*

4. *For all $n, m \in \mathsf{N}$, $\tan(m\pi/n)$ is algebraic. (This follows from parts 2,3 and the relationship between tangent, sine, cosine, and the closure of the algebraic numbers under quotients.)*

Proof:

Let $n \in \mathsf{N}$.
As noted above,

$$T_n(\cos(n\theta) = \cos(\theta).$$

Let $\theta = \frac{\pi}{n}$. Then you get

$$T_n(\cos(\frac{\pi}{n}) = \cos(\pi) = -1.$$

On the right hand side you get a poly in $\cos(\frac{\pi}{n})$ with integer coefficients. Hence $\cos(\frac{\pi}{n})$ is algebraic. ∎

References

[1] H. Boas. The oldest trig in the book. *College Mathematics Journal*, 50(1):9–20, 2019.