Standard and NonStandard Dice: An Exposition

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If you roll two standard 6-sided dice then

1. 2: (1,1). ONE way. Prob $\frac{1}{36}$.
2. 3: (1,2), (2,1). TWO ways. Prob $\frac{1}{18}$.
3. 4: (1,3), (2,2), (3,1). THREE ways. Prob $\frac{1}{12}$.
4. 5: (1,4), (2,3), (3,2), (4,1). FOUR ways. Prob $\frac{1}{9}$.
5. 6: (1,5), (2,4), (3,3), (4,2), (5,1) FIVE ways. Prob $\frac{5}{36}$.
6. 7: (1,6), (2,5), (3,4), (4,3) ,(5,2) ,(6,1) SIX ways. Prob $\frac{1}{6}$.
7. 8: (2,6), (3,5), (4,4), (5,3), (6,2) FIVE ways. Prob $\frac{5}{36}$.
8. 9: (3,6), (4,5), (5,4), (6,3) FOUR ways. Prob $\frac{1}{9}$.
9. 10: (4,6), (5,5),(6,4) THREE ways. Prob $\frac{1}{12}$.
10. 11: (5,6), (6,5) TWO ways. Prob $\frac{1}{18}$.
11. 12: (6,6) ONE way. Prob $\frac{1}{36}$.

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Coefficient of x^n is number of ways to get n.

Example of Non-Standard Labelings

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1. 12: TWO ways. Prob $\frac{1}{18}$.
2. 9: THREE ways. Prob $\frac{1}{12}$.
3. 8: NINE ways. Prob $\frac{1}{4}$.
4. 6: TWO ways. Prob $\frac{1}{18}$.
5. 5: TWELVE ways. Prob $\frac{1}{3}$.
6. 4: FOUR ways. Prob $\frac{1}{9}$.
7. 3: THREE ways. Prob $\frac{1}{12}$.
8. 2: ONE ways. Prob $\frac{1}{36}$.

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$$(x^{a_1}+x^{a_2}+x^{a_3}+x^{a_4}+x^{a_5}+x^{a_6})(x^{b_1}+x^{b_2}+x^{b_3}+x^{b_4}+x^{b_5}+x^{b_6}) =$$

$$(x^6+x^5+x^4+x^3+x^2+x)^2.$$

Is there a Non-Standard Labeling That... Cont.

$$(x^{a_1}+x^{a_2}+x^{a_3}+x^{a_4}+x^{a_5}+x^{a_6})(x^{b_1}+x^{b_2}+x^{b_3}+x^{b_4}+x^{b_5}+x^{b_6}) =$$

$$(x^6+x^5+x^4+x^3+x^2+x)^2 = x^2(x^5+x^4+x^3+x^2+x+1)^2 =$$

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

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What properties do the polys we are looking for have?

1. $a_6 = 1$ and $b_6 = 1$ since otherwise cannot get a 2. So both poly's have a factor of x.
2. $(1^{a_1} + 1^{a_2} + x^{a_3} + 1^{a_4} + 1^{a_5} + 1^{a_6}) = 6$. So if $f(x)$ is a factor need $f(1) = 6$.

Is there a Non-Standard Labeling That... Cont.

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2 =$$

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Since $f(1) = 6$ and $g(1) = 6$ we have conditions
$1 \times 2^a \times 1^b \times 3^c = 6$ and $1 \times 2^{2-a} \times 1^{2-b} \times 3^{2-c} = 6$.
So

$$a = 1 \qquad b \in \{0, 1, 2\} \qquad c = 1.$$

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So

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$b = 0$ and $b = 2$ are symmetric so we just do $b = 0$ and $b = 1$.

The Non-Standard Labeling

Case $b = 0$: Then the polynomials for the dice are
$x(x+1)(x^2+x+1) = x^4+2x^3+2x^2+x$.
$x(x+1)(x^2-x+1)^2(x^2+x+1) = x^8+x^6+x^5+x^4+x^3+x$.

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Case $b = 1$: Then the polynomials for the dice are
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So the dice are $(1,2,3,4,5,6)$ and $(1,2,3,4,5,6)$.
The standard dice.

Upshot there is only ONE pair of nonstandard dice that give the same probabilities as the standard dice. That pair is
$(1,2,2,3,3,4)$ and $(1,3,4,5,6,8)$.