

1 A Muffin-Theorem Generator

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9th International Conference on Fun with Algorithms (FUN 2018).

Editors: Hiro Ito, Stephano Leonardi, Linda Pagli, and Giuseppe Prencipe; Article No. 15; pp. 15:1–15:19



Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

38 — Abstract —

39 Consider the following FUN problem. Given m, s you want to divide m muffins among s students
40 so that everyone gets $\frac{m}{s}$ muffins; however, you want to maximize the minimum piece so that
41 nobody gets crumbs. Let $f(m, s)$ be the size of the smallest piece in an optimal procedure.

42 We study the case where $\lceil \frac{2m}{s} \rceil = 3$ because (1) many of our hardest open problems were of
43 this form until we found this method, (2) we have used the technique to *generate muffin-theorems*,
44 and (3) we conjecture this can be used to solve the general case. We give (1) an algorithm to
45 find an upper bound for $f(m, s)$ when $\lceil \frac{2m}{s} \rceil = 3$ (and some ways to speed up that algorithm if
46 certain conjectures are true), (2) an algorithm that uses the information from (1) to try to find
47 a lower bound on $f(m, s)$ (a procedure) which matches the upper bound, (3) an algorithm that
48 uses the information from (1) to generate muffin-theorems, and (4) an algorithm that we think
49 works well in practice to find $f(m, s)$ for any m, s .

50 **2012 ACM Subject Classification** Mathematics of Computing → Combinatorial Optimization

51 **Keywords and phrases** Fair Division, Theorem Generation

52 **Digital Object Identifier** 10.4230/LIPIcs.FUN.2018.15

53 **Related Version** <https://arxiv.org/abs/1709.02452>

54 **Acknowledgements** We thank Nancy Blachman who compiled the list of problems that intro-
55 duced us to this problem, Alan Frank who first came up with the problem. We would also like
56 to thank Alan Frank, James Propp, and Sam Zbarsky for stimulating discussions of the topic.

57 **1** Introduction

58 Consider the following FUN problem. Given m, s you want to divide m muffins among s
59 students so that everyone gets $\frac{m}{s}$ muffins; however, you want to maximize the minimum
60 piece so that nobody gets crumbs. Let $f(m, s)$ be the size of the smallest piece in an optimal
61 procedure.

62 We give an example:

63 *You have 47 muffins and 36 students. You want to divide the muffins evenly, but no student*
64 *wants a small piece. Find a protocol that maximizes the smallest piece. We show in Section 5*
65 *that there is a procedure for this with smallest piece $\frac{31}{90}$ and that this is optimal. Hence*
66 *$f(47, 36) = \frac{31}{90}$.*

67 **Convention** When discussing a muffin being cut we refer to *pieces*. When discussing a
68 student receiving we refer to *shares*. They are the same; however, it will be good to have
69 different terminologies to focus on what's important. We treat a piece, a share, and its value
70 as the same thing. So we may say let $x \geq \frac{1}{3}$ be given to a student.

71 **► Definition 1.** Let $m, s \in \mathbb{N}$. An (m, s) -protocol is a protocol to cut m muffins into pieces
72 and then distribute them to the s students so that each student gets $\frac{m}{s}$ muffins. An (m, s) -
73 protocol is *optimal* if it has the largest smallest piece of any protocol. $f(m, s)$ is the size of
74 the smallest piece in an optimal (m, s) -protocol.

75 Clearly, for all $a \in \mathbb{N}$, $f(am, as) \geq f(m, s)$. All of our theorems indicate that $f(am, as) =$
76 $f(m, s)$. We have not been able to prove this; however, we will only consider the cases where
77 m, s are relatively prime.

We came upon this problem in a pamphlet *Julia Robinson Mathematics Festival: A Sample of Mathematical Puzzles* compiled by Nancy Blachman. On Page 2 was *The Muffin Puzzle* which asked about the problem for several particular cases. Nancy Blachman attributes the problem to Alan Frank and points out that it was described by Jeremy Copeland [3]. We are the first ones to consider this problem seriously for general m, s with one caveat: There was some discussion of this problem in the math-fun email list in 2009. We have obtained a copy of their archives and discovered that they already had Theorem 3 and 11. We will credit the individuals when we get to those theorems.

Given m, s how hard is it to compute $f(m, s)$? Computing $f(m, s)$ can be rephrased as a mixed integer program on $O(ms)$ variables (the proof is in the Section A). Since the input is of size $O(\log m + \log s)$ this result does not even put the problem into NP. One of the upshots of this paper will be a procedure that we conjecture puts the computation of $f(m, s)$ into P.

We study the case where $\lceil \frac{2m}{s} \rceil = 3$ because (1) many of our hardest open problems were of this form until we found this method, (2) we have used the technique to *generate muffin-theorems*, (3) we conjecture this can be used to solve the general case.

We have a long paper [2] and some programs [1] for computing $f(m, s)$. For $1 \leq s \leq 50$, $1 \leq m \leq 60$ we have computed $f(m, s)$. In this paper we focus on a subset of the material that lends itself to generating theorems about muffins via an algorithm.

2 Summary of Results

In Sections 3,4 we give basic theorems and definitions used throughout the paper. In Section 5 we illustrate the *Buddy-Match techniques* by proving $f(47, 36) \leq \frac{31}{90}$. In Section 6 we illustrate how to obtain lower bounds and present the result $f(47, 36) \geq \frac{31}{90}$.

In Sections 7 we discuss how to generate theorems from the Buddy-Match Technique. These theorems are of the form:

If $d \in \mathbb{N}$ and $1 \leq a \leq 3d - 1$, a, d relatively primes, then

$$(\forall k \geq 1) \left[f(3dk + a + d, 3dk + a) \leq \frac{dk + X}{3dk + a} \right]$$

where X is a constant which can depend on a, d but not on k . In Section 8 we discuss how to generate theorems that are more general. Here is an example:

If $1 \leq a \leq \frac{5d}{7}$ and $a \neq \frac{2d}{3}$ then $f(3dk + a + d, 3dk + a) \leq \frac{dk + X}{3dk + a}$ where $X = \max\{\frac{2a}{5}, \frac{a+d}{6}\}$.

In Sections 10, 11 we show how, assuming certain conjectures, one can speed up the Buddy-Match Technique. In Section 12 we give an algorithm that we conjecture puts $f(m, s)$ into P. In Section 13 we speculate about that algorithm and other muffin-issues.

In the appendix we state and sometimes prove theorems that are needed to fill in some of the gaps in our narrative. We also give some examples of the theorems we generated.

3 Basic Theorems

In this section we prove two theorems that will enable us, for the rest of the paper, to only consider m, s and protocols such that (1) $m > s \geq 3$, (2) s does not divide m , and (4) every muffin is cut into exactly two pieces.

The following theorem takes care of the cases $s = 1$ and $s = 2$. The proofs are easy and left to the reader.

117 ► **Theorem 2.**

- 118 1. $(\forall m)[f(m, 1) = 1]$
 119 2. $(\forall m)[m \equiv 0 \pmod{2} \rightarrow f(m, 2) = 1]$
 120 3. $(\forall m)[m \equiv 1 \pmod{2} \rightarrow f(m, 2) = \frac{1}{2}]$
 121 4. $(\forall m, s)[s \text{ divides } m \rightarrow f(m, s) = 1]$.

122 The following theorem shows that if you know $f(m, s)$ then you know $f(s, m)$. Combined
 123 with Theorem 2 we need only consider $m > s \geq 3$. This theorem was independently discovered
 124 by Erich Friedman, within the math-fun email list, in 2009.

125 ► **Theorem 3.** *Let $m, s \in \mathbb{N}$. Then $f(s, m) = \frac{s}{m}f(m, s)$.*

126 **Proof.** Assume $f(m, s) \geq \alpha$. We show $f(s, m) \geq \frac{s}{m}\alpha$. Let $M_1, \dots, M_m$ be the muffins. Let
 127 $S_1, \dots, S_s$ be the students. The protocol that achieves $f(m, s) \geq \alpha$ must be of the following
 128 form:

- 129 1. For each $1 \leq i \leq m$ divide M_i into pieces $(a_{i1}, a_{i2}, \dots, a_{im_i})$ where $\sum_{j=1}^{m_i} a_{ij} = 1$.
 130 2. For each $1 \leq j \leq s$ give S_j the shares $[b_{1j}, b_{2j}, \dots, b_{s_jj}]$ where $\sum_{i=1}^{s_j} b_{ij} = \frac{m}{s}$.

131 The following hold:

- 132 ■ $\bigcup_{i=1}^m \bigcup_{j=1}^{m_i} \{a_{ij}\} = \bigcup_{j=1}^s \bigcup_{i=1}^{s_j} \{b_{ij}\}$
 133 ■ The min over all of the a_{ij} is α .

134 The following protocol shows that $f(s, m) \geq \frac{s}{m}\alpha$. Let $M'_1, \dots, M'_s$ be the muffins. Let
 135 $S'_1, \dots, S'_m$ be the students.

- 136 1. For each $1 \leq j \leq s$ divide M'_j into $(\frac{s}{m}b_{1j}, \frac{s}{m}b_{2j}, \dots, \frac{s}{m}b_{s_jj})$. Note that $\sum_{i=1}^{s_j} \frac{s}{m}b_{ij} =$
 137 $\frac{s}{m} \sum_{i=1}^{s_j} b_{ij} = \frac{s}{m} \times \frac{m}{s} = 1$.
 138 2. For each $1 \leq i \leq m$ give S'_i $[\frac{s}{m}a_{i1}, \frac{s}{m}a_{i2}, \dots, \frac{s}{m}a_{im_i}]$. Note that $\sum_{j=1}^{m_i} \frac{s}{m}a_{ij} = \frac{s}{m} \sum_{j=1}^{m_i} a_{ij} =$
 139 $\frac{s}{m} \times 1 = \frac{s}{m}$.

140 Clearly this is a correct protocol and the minimum piece is of size $\frac{s}{m}\alpha$.

141 We now show that $f(s, m) = \frac{s}{m}f(m, s)$. By the above we have both (1) $f(s, m) \geq$
 142 $\frac{s}{m}f(m, s)$, and (2) $f(m, s) \geq \frac{m}{s}f(s, m)$. Hence

$$f(s, m) \geq \frac{s}{m}f(m, s) \geq \frac{s}{m} \frac{m}{s} f(s, m) = f(s, m).$$

143 Therefore $f(s, m) = \frac{s}{m}f(m, s)$. ◀

144 ► **Theorem 4.** *Let $m, s \in \mathbb{N}$.*

- 145 1. *If $f(m, s) \geq \alpha$ and $\alpha > \frac{1}{3}$ via protocol P then protocol P cuts every muffin into 1 or 2*
 146 *pieces.*
 147 2. *$f(m, s) \geq \alpha$ and $\alpha \leq \frac{1}{2}$ via protocol P then there is a protocol P' such that (1) P' also*
 148 *yields $f(m, s) \geq \alpha$, and (2) P' cuts every muffin into 2 or more pieces.*

149 **Proof.** a) If any muffin is cut into ≥ 3 pieces then there is a piece $\leq \frac{1}{3} < \alpha$.

150 b) If any muffin is uncut and given to (say) Alice then we can add a step where we cut the
 151 muffin into $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$ -sized pieces to Alice. Since $\alpha \leq \frac{1}{2}$ adding in some pieces
 152 of size $\frac{1}{2}$ does not affect the smallest piece. ◀

153 By Theorem 4 we have the following convention.

154 **Convention:** When trying to show that $f(m, s) \leq \alpha$ where $\frac{1}{3} < \alpha < \frac{1}{2}$ we will assume, by
 155 way of contradiction, that there is a protocol showing $f(m, s) > \alpha$ where every muffin is cut
 156 into exactly 2 pieces.

4 Basic Definitions

► **Definition 5.** Let $m, s \in \mathbb{N}$. Assume there is an (m, s) -protocol.

1. The two pieces that come from the same muffin are called *buddies*. $B(x)$ is the buddy of x . Note that $B(x) = 1 - x$.
2. A student that gets A shares is an A -student. A share given to an A -student is an A -share.
3. 2-Shares that are given to the same 2-student are *matched*. $M(x)$ is the match of 2-share x . Note that $M(x) = \frac{m}{s} - x$.
4. If x is a share given to a 3-student then $M_S(x)$ is the smallest share (not including x) that the student has, and $M_L(x)$ is the largest. Note that $M_S(x) \leq \frac{(m/s)-x}{2}$. Hence $B(M_S(x)) \geq 1 - \frac{(m/s)-x}{2}$.

Notation: (a, b) will mean the set of shares that have size strictly between a and b . Hence $|(a, b)|$ will be the number of such shares. We use similar notation for $[a, b]$.

5 An Example is Worth A Thousand Theorems: 43 muffins, 39 Students

The method we demonstrate in this section is called *The Buddy-Match Method*.

► **Theorem 6.** $f(47, 36) \leq \frac{31}{90} = \frac{124}{360}$.

Proof. To make the notation easier we write all fractions as having denominator 360.

Assume there is an $(47, 36)$ -procedure. We show that there is a piece $\leq \frac{124}{360}$. Note that $\frac{47}{36} = \frac{470}{360}$.

Case 1: Some student gets ≥ 4 shares. Then some students has a share $\leq \frac{47}{36 \times 4} < \frac{124}{360}$.

Case 2: Some student gets ≤ 1 share. $1 < \frac{47}{36}$, so this is impossible.

Case 3: Every muffin is cut in 2 pieces and every student gets either 2 or 3 shares. The total number of shares is 94. Let s_2 (s_3) be the number of 2-students (3-students).

$$2s_2 + 3s_3 = 94$$

$$s_2 + s_3 = 36$$

So $s_2 = 14$ and $s_3 = 22$.

Case 3.1: There is a 2-share $x \leq \frac{234}{360}$. $M(x) \geq \frac{470}{360} - \frac{234}{360} = \frac{236}{360}$ so $B(M(x)) \leq 1 - \frac{236}{360} = \frac{124}{360}$

Case 3.2: There is a 3-share $x \geq \frac{222}{360}$. $B(M_S(x)) \leq 1 - \frac{\frac{470}{360} - \frac{222}{360}}{2} = \frac{124}{360}$.

Case 3.3: There is a 2-share $x \geq \frac{236}{360}$. $B(x) \leq 1 - \frac{236}{360} = \frac{124}{360}$

Case 3.4: There is a 3-share $x \leq \frac{124}{360}$. This one is self-explanatory.

Case 3.5: All 3-shares are in $(\frac{124}{360}, \frac{222}{360})$ and all 2-shares are in $(\frac{234}{360}, \frac{236}{360})$.

The following picture captures what we know so far.

$$\left(\frac{124}{360} \quad \text{---} \quad \frac{222}{360} \right) \text{ 3-shs } \left(\frac{234}{360} \quad \text{---} \quad \frac{236}{360} \right) \text{ 2-shs }$$

Since there are no shares in $[\frac{222}{360}, \frac{234}{360}]$, there are no shares in $B([\frac{222}{360}, \frac{234}{360}]) = [\frac{126}{360}, \frac{138}{360}]$

The following picture captures what we know so far.

$$\left(\frac{124}{360} \quad \text{---} \quad \frac{126}{360} \right) \text{ S3-shs } \left(\frac{138}{360} \quad \text{---} \quad \frac{222}{360} \right) \text{ L3-shs } \left(\frac{234}{360} \quad \text{---} \quad \frac{236}{360} \right) \text{ 2-shs }$$

192 S3-shs stands for *short 3-shares* and L3-shs stands for *large 3-shares*. There are $2s_2 = 28$
 193 2-shares so there are 28 S3-shares (B is a bijection between 2-shares and S3-shares). Since
 194 there are $3s_3 = 66$ 3-shares total that leaves 38 S3 shares.

195 Since the midpoint of L3-shs is $\frac{360}{2}$, the Buddy function is a bijection from $(\frac{138}{360}, \frac{180}{360})$ to
 196 $(\frac{180}{360}, \frac{222}{360})$. Hence these two intervals have the same number of shares.

197 Since the midpoint of 2-shs is $\frac{470}{2}$, the Match function is a bijection from $(\frac{234}{360}, \frac{235}{360})$ to
 198 $(\frac{235}{360}, \frac{236}{360})$. Hence these two intervals have the same number of shares. Applying the Buddy
 199 function to both these intervals we obtain that $(\frac{124}{360}, \frac{125}{360})$ and $(\frac{125}{360}, \frac{126}{360})$ have the same number
 200 of shares.

201 In the scenarios above there are an even number of shares of size the midpoint. We
 202 arbitrarily assign half to the left and half to the right.

203 We define the following intervals.

204 ► **Definition 7.**

- 205 1. $I_1 = (\frac{124}{360}, \frac{125}{360})$
- 206 2. $I_2 = (\frac{125}{360}, \frac{126}{360})$ ($|I_1| = |I_2|$, $|I_1 \cup I_2| = 28$)
- 207 3. $I_3 = (\frac{138}{360}, \frac{180}{360})$
- 208 4. $I_4 = (\frac{180}{360}, \frac{222}{360})$ ($|I_3| = |I_4|$, $|I_3 \cup I_4| = 38$)

209 Henceforth all of the students considered will be 3-students. We now look at the students
 210 in a more detailed way than 2-students and 3-students.

211 ► **Definition 8.** Let $1 \leq i_1 \leq \dots \leq i_3 \leq 4$. An $e(i_1, i_2, i_3)$ -student is a student who has, for
 212 each $1 \leq j \leq 3$, a share in I_{i_j} . For example, an $e(1, 1, 4)$ -students has two shares in I_1 and
 213 one share in I_4 .

214 **Claim 1:**

- 215 1. The only possible students are:

- 216 a. $e(1, 1, 4)$
- 217 b. $e(1, 2, 4)$
- 218 c. $e(1, 3, 3)$
- 219 d. $e(1, 3, 4)$
- 220 e. $e(2, 2, 4)$
- 221 f. $e(2, 3, 3)$
- 222 g. $e(2, 3, 4)$
- 223 h. $e(3, 3, 3)$
- 224 i. $e(3, 3, 4)$

- 225 2. There are no shares in $[\frac{208}{360}, \frac{218}{360}]$

- 226 3. There are no shares in $[\frac{142}{360}, \frac{152}{360}]$ (this follows from the prior part and buddying).

227 **Proof of Claim 1:**

- 228 1) We establish that some students are impossible.

229 A $e(1, 4, 4)$ -student has more than $\frac{124}{360} + 2 \times \frac{180}{360} = \frac{484}{360}$

230 A $e(2, 2, 3)$ -student has less than $2 \times \frac{126}{360} + \frac{180}{360} = \frac{432}{360}$

231 The result follows from these two statements, though the proof is tedious.

- 232 2) We look at which I_4 -shares are used

233 A $e(1, 1, 4)$ student uses I_4 -share $> \frac{470}{360} - 2 \times \frac{125}{360} = \frac{220}{360}$

234 A $e(1, 2, 4)$ student uses I_4 -shares $> \frac{470}{360} - \frac{125}{360} - \frac{126}{360} = \frac{219}{360}$

235 A $e(1, 3, 4)$ student uses I_4 -shares $< \frac{470}{360} - \frac{124}{360} - \frac{138}{360} = \frac{208}{360}$

236 A $e(2, 2, 4)$ student uses I_4 -shares $> \frac{470}{360} - 2 \times \frac{126}{360} = \frac{218}{360}$

237 A $e(2, 3, 4)$ student uses I_4 -shares $< \frac{470}{360} - \frac{125}{360} - \frac{138}{360} = \frac{207}{360}$

238 A $e(3, 3, 4)$ student uses I_4 -shares $< \frac{470}{360} - 2 \times \frac{138}{360} = \frac{194}{360}$
 239 Hence the only shares in I_4 that can be used are those $< \frac{208}{360}$ or $> \frac{218}{360}$. The result follows.

240 **End of Proof of Claim 1**

241 We redefine the intervals.

242 **► Definition 9.**

- 243 1. $I_1 = (\frac{124}{360}, \frac{125}{360})$
- 244 2. $I_2 = (\frac{125}{360}, \frac{126}{360})$ ($|I_1| = |I_2|$, $|I_1 \cup I_2| = 28$)
- 245 3. $I_3 = (\frac{138}{360}, \frac{142}{360})$
- 246 4. $I_4 = (\frac{152}{360}, \frac{180}{360})$
- 247 5. $I_5 = (\frac{180}{360}, \frac{208}{360})$ ($|I_4| = |I_5|$)
- 248 6. $I_6 = (\frac{218}{360}, \frac{222}{360})$ ($|I_3| = |I_6|$, $|I_3 \cup I_4 \cup I_5 \cup I_6| = 38$)

249 By a proof similar to that of Claim 1 we obtain the following:

250 **Claim 2:**

- 251 1. The only possible students are: $e(1, 1, 6)$, $e(1, 2, 6)$, $e(1, 3, 5)$, $e(1, 4, 4)$, $e(1, 4, 5)$, $e(2, 2, 6)$,
 252 $e(2, 3, 5)$, $e(2, 4, 4)$, $e(2, 4, 5)$, $e(3, 3, 5)$, $e(3, 4, 4)$, and $e(4, 4, 4)$.
- 253 2. There are no shares in $[\frac{194}{360}, \frac{202}{360}]$
- 254 3. There are no shares in $[\frac{158}{360}, \frac{166}{360}]$ (this follows from the prior part and buddying).

255 We define the following intervals.

256 **► Definition 10.**

- 257 1. $I_1 = (\frac{124}{360}, \frac{125}{360})$
- 258 2. $I_2 = (\frac{125}{360}, \frac{126}{360})$ ($|I_1| = |I_2|$, $|I_1 \cup I_2| = 28$)
- 259 3. $I_3 = (\frac{138}{360}, \frac{142}{360})$
- 260 4. $I_4 = (\frac{152}{360}, \frac{158}{360})$
- 261 5. $I_5 = (\frac{166}{360}, \frac{180}{360})$
- 262 6. $I_6 = (\frac{180}{360}, \frac{194}{360})$ ($|I_5| = |I_6|$)
- 263 7. $I_7 = (\frac{202}{360}, \frac{208}{360})$ ($|I_4| = |I_7|$)
- 264 8. $I_8 = (\frac{218}{360}, \frac{222}{360})$ ($|I_3| = |I_8|$, $|I_3 \cup \dots \cup I_8| = 38$)

265 By a proof similar to that of Claim 1 we obtain:

266 **Claim 3:** The only possible students are: $e(1, 1, 8)$, $e(1, 2, 8)$, $e(1, 3, 7)$, $e(1, 4, 6)$, $e(1, 5, 5)$,
 267 $e(2, 2, 8)$, $e(2, 3, 7)$, $e(2, 4, 6)$, $e(2, 5, 5)$, $e(3, 3, 6)$, and $e(4, 4, 4)$.

268 Let

- 269 1. $|e(1, 1, 8)| = a$
- 270 2. $|e(1, 2, 8)| = b$
- 271 3. $|e(1, 3, 7)| = c$
- 272 4. $|e(1, 4, 6)| = d$
- 273 5. $|e(1, 5, 5)| = e$
- 274 6. $|e(2, 2, 8)| = f$
- 275 7. $|e(2, 3, 7)| = g$
- 276 8. $|e(2, 4, 6)| = h$
- 277 9. $|e(2, 5, 5)| = i$
- 278 10. $|e(3, 3, 6)| = j$
- 279 11. $|e(4, 4, 4)| = k$

280 Since $|I_1| = |I_2|$, $2a + b + c + d + e = b + 2f + g + h + i$, so $2a + c + d + e = 2f + g + h + i$
 281 Since $|I_3| = |I_8|$, $c + g + 2j = a + b + f$
 282 Since $|I_4| = |I_7|$, $d + h + 3k = c + g$
 283 Since $|I_5| = |I_6|$, $2e + 2i = d + h + j$
 284 Since $|I_1 \cup I_2| = 28$, $2a + 2b + c + d + e + 2f + g + h + i = 28$
 285 Since there are 22 3-students, $a + b + c + d + e + f + g + h + i + k = 22$
 286 From the last two equations we obtain $a + b + f = 6$

We combine I_1 and I_2 into a single interval. This reduces the system to 6 variables, resulting in the equation

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 22 \\ 28 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

287 However, one can check that eliminating the bottom 3 rows requires the top 2 rows to be in
 288 the ratio $7 : 9$. $22 : 28 \neq 7 : 9$, so there is no solution. ◀

289 The above proof used that $\lceil \frac{2m}{s} \rceil = 3$ since that is the condition that leads to having
 290 2-shares and 3-shares. This is usually important since it gives us symmetry from matches, not
 291 just from buddying; however, in this case we just so happened to not need that symmetry.

292 6 Finding a Procedure

293 We now describe the program that finds the procedure showing $f(47, 36) \geq \frac{124}{360}$. We *guess*
 294 that all shares are of the form $\frac{x}{360}$ where $124 \leq x \leq 236$. But we can cut down those variables
 295 a *lot* based on the proof. For example, by modifying the proof slightly, we can deduce that
 296 there are no share of size $\frac{127}{360}, \frac{128}{360}, \dots, \frac{137}{360}$. This is a key factor in speeding up the program.
 297 We can also use the symmetries of where shares can be.

298 For every way to split a muffin we have a variable for how many muffins are split that
 299 way, as follows: $(\frac{124}{360}, \frac{236}{360})$ is associated to the variables $y_{124,236}$, $(\frac{125}{360}, \frac{235}{360})$ is associated with
 300 the variable $y_{125,235}$, etc. This variable is the *number of muffins* that are split that way.

301 For every way to give muffin shares to a student we have a variable for how many students
 302 get that set of shares, as follows: $[\frac{87}{360}, \frac{79}{360}, \frac{69}{360}]$ is associated to the variable $z_{87,79,69}$, $[\frac{118}{360}, \frac{117}{360}]$
 303 is associated to the variables $z_{118,117}$, etc. This variable is the *number of students* who get
 304 that share-size.

305 For each size we express how many pieces are of that size in two ways.

- 306 ■ The number of pieces of that size based on the muffins. For example, the number of
 307 pieces of size $\frac{131}{360}$ is $y_{131,256}$. The number of pieces of size $\frac{180}{360}$ is $2 \times y_{180,180}$.
- The number of shares of that size based on the students. For example, the number of
 shares of size $\frac{131}{360}$ is

$$z_{124,131,215} + \dots + z_{130,131,209} + 2z_{131,131,208} + z_{132,131,207} + \dots + z_{215,131,124}$$

308 For each size we get an equation by equating the muffin-based and student-based ex-
 309 pressions. We have more equations based on the number of pieces and the number in each
 310 interval which falls out of the proof of the upper bound. This leads to a set of linear equations
 311 whose solution leads to a procedure.

312 Here is the procedure for $f(47, 36) \geq \frac{124}{360} = \frac{117}{180}$ we obtained with this method:

- 313 1. Divide 1 muffin ($\frac{90}{180}, \frac{90}{180}$)
- 314 2. Divide 2 muffins ($\frac{93}{180}, \frac{87}{180}$)
- 315 3. Divide 2 muffins ($\frac{101}{180}, \frac{79}{180}$)
- 316 4. Divide 2 muffins ($\frac{104}{180}, \frac{76}{180}$)
- 317 5. Divide 6 muffins ($\frac{109}{180}, \frac{71}{180}$)
- 318 6. Divide 6 muffins ($\frac{111}{180}, \frac{69}{180}$)
- 319 7. Divide 14 muffins ($\frac{117}{180}, \frac{63}{180}$)
- 320 8. Divide 14 muffins ($\frac{118}{180}, \frac{62}{180}$)
- 321 9. Give 2 students $[\frac{87}{180}, \frac{79}{180}, \frac{69}{180}]$
- 322 10. Give 2 students $[\frac{90}{180}, \frac{76}{180}, \frac{69}{180}]$
- 323 11. Give 2 students $[\frac{93}{180}, \frac{71}{180}, \frac{71}{180}]$
- 324 12. Give 2 students $[\frac{101}{180}, \frac{71}{180}, \frac{63}{180}]$
- 325 13. Give 2 students $[\frac{104}{180}, \frac{69}{180}, \frac{62}{180}]$
- 326 14. Give 6 students $[\frac{109}{180}, \frac{63}{180}, \frac{63}{180}]$
- 327 15. Give 6 students $[\frac{111}{180}, \frac{62}{180}, \frac{62}{180}]$
- 328 16. Give 14 students $[\frac{118}{180}, \frac{117}{180}]$

329 The reader should be able to see how to generalize the method outlined above.

330 What is described above is not quite what we have coded up (though we will). The
 331 Interval Method (see Section B) is another method to find lower bounds that gives information
 332 that can be used to cut down the time to find a procedure. We have coded up a version of
 333 what is outlined above with the interval method.

334 We denote the algorithm given above (the one using Buddy-Match) $VLOWER(m, s, \alpha)$
 335 where one finds a procedure showing $f(m, s) \geq \alpha$, hence verifying that $f(m, s) \geq \alpha$.

336 7 The Proof that $f(47, 36) \leq \frac{31}{90}$ Reveals Much More

337 The proof that $f(47, 36) \leq \frac{31}{90}$ can be modified very slightly (just notation) to obtain the
 338 following result (which we write in a strange way for later exposition):

$$(\forall k \geq 1) \left[f(3 \times 11 \times k + 11 + 3, 33k + 3) \leq \frac{11k + \frac{7}{5}}{3 \times 11 \times 3k + 3} \right]$$

339 More generally the following seems to be true empirically:

340 *for all d (d stands for difference and is $m - s$), for all $1 \leq a \leq 3d - 1$ (a, d relatively primes),*
 341 *there exists X :*

$$(\forall k \geq 1) \left[f(3dk + d + a, 3dk + a) \leq \frac{dk + X}{3dk + a} \right]$$

342
 343 For $d = 1$ to 8, for all relevant a , we have found X . In many concrete cases we have
 344 shown that it is also an upper bound. In Section C we present the results for the $d = 7$ case.

345 Note that we need $k \geq 1$ since if $k = 0$ then we no longer have $\lceil \frac{2m}{s} \rceil = 3$.

346 8 Generating More General Theorems

347 The techniques discussed in Section 7 generate theorems of the form

$$(\forall k \geq 1) \left[f(3dk + a + d, 3dk + a) \leq \frac{dk + X}{3dk + a} \right].$$

348 However, the program can be modified to obtain more general theorems. As noted in
 349 Section 7 our program finds interesting values of X . That is, the program may find that
 350 (say) if $X \leq \frac{7}{6}$ then there are no $e(1, 3, 4)$ -students. What is it about $X \leq \frac{7}{6}$ that makes this
 351 happen? It may be that (say) $1 \leq a \leq \frac{5d}{7}$ and $a \neq \frac{2d}{3}$ makes this work, and it may be that
 352 $X = \max\{\frac{2a}{5}, \frac{a+d}{6}\}$.

353 We have taken the results from the program and, with the help of additional programs
 354 and our own ingenuity generated many theorems (we hope to fully automate it soon). These
 355 theorems are a great time saver since often the result we want falls out of them directly. We
 356 present a sample of such theorems in the Section D.

357 9 How to find X

358 The proof of Theorem 6 can be summarized as follows: The assumption $f(47, 36) > \frac{31}{90}$
 359 implies that a certain system of linear equations have a solution where all of the variables
 360 are natural numbers between 0 and $s_3 = 22$. The system had no such solution, hence a
 361 contradiction.

Imagine that we want an upper bound on $f(47, 36)$ but do not know what it is ahead of
 time. Following the line of reasoning in Section 7 we seek X such that

$$f(33 + 3 + 11, 33 + 3) \leq \frac{11 + X}{33 + 3}.$$

362 We use a program to simulate the proof of Theorem 6 but with X instead of the actual
 363 numbers. This program will produce many values of X where something interesting happens,
 364 such as a type of student no longer being allowed. The program looks at the (finite) set of
 365 interesting values of X and finds the least one that causes the resulting system of linear
 366 equations to be unsolvable using natural numbers between 0 and 22. Hence we have a value
 367 of X . We then use $VLOWER(47, 36, \frac{11+X}{36})$ to find the matching lower bound (if this does
 368 not work then the algorithm failed to find $f(m, s)$).

369 For the values 47, 36 it was easy to find the value of X . For larger m, s it may be that
 370 verifying $f(m, s) \leq \alpha$ is faster than finding the α . In the next two sections we examine how
 371 to speed up finding X .

372 We leave it to the reader to generalize the algorithm to any m, s where $\lceil \frac{2m}{s} \rceil = 3$; however,
 373 we give the following picture which represents intervals where 3-shares can be. In the picture
 374 each nonempty interval has the number of 3-shares in it (though y is not known) and a label
 375 such as I_1 so we can refer to it. This picture is the result of many buddy-match sequences.

$$\begin{array}{c}
 376 \quad \left(\begin{array}{c} a + d \quad (I_1) \\ \frac{dk+X}{3dk+a} \end{array} \mid \begin{array}{c} a + d \quad (I_2) \\ \frac{dk+\frac{a}{2}}{3dk+a} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{dk+a-X}{3dk+a} \end{array} \mid \begin{array}{c} 0 \\ \frac{dk+2X}{3dk+a} \end{array} \right] \\
 377 \quad \left(\begin{array}{c} y \quad (I_3) \\ \frac{dk+2X}{3dk+a} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{dk+a+d-3X}{3dk+a} \end{array} \mid \begin{array}{c} 0 \\ \frac{dk+d-a+2X}{3dk+a} \end{array} \right] \\
 378 \quad \left(\begin{array}{c} 2d - a - y \quad (I_4) \\ \frac{dk+d-a+2X}{3dk+a} \end{array} \mid \begin{array}{c} 2d - a - y \quad (I_5) \\ \frac{dk+\frac{a+d}{2}}{3dk+a} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{dk+2a-2X}{3dk+a} \end{array} \mid \begin{array}{c} y \quad (I_6) \\ \frac{dk+3X}{3dk+a} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{dk+a+d-2X}{3dk+a} \end{array} \right] \\
 379 \quad \left(\begin{array}{c} y \quad (I_6) \\ \frac{dk+3X}{3dk+a} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{dk+a+d-2X}{3dk+a} \end{array} \right]
 \end{array}$$

380 Facts and Caveats:

- 381 1. $|I_1| = |I_2|$
- 382 2. $|I_4| = |I_5|$
- 383 3. In the picture it is unclear if the endpoint of I_1 is included in I_1 . We do not include
- 384 it; however, we take the even number of shares that are at that endpoint and arbitrary
- 385 assign half to I_1 and half to I_2 .
- 386 4. There is a similar comment for I_2 , I_4 , and I_5 .

387 We denote the version where you do not already have upper bound to check $\text{BUDMAT}(m, s)$
 388 and the version where you do $\text{BUDMAT}(m, s, \alpha)$ where α is the bound. We will avoid using
 389 $\text{BUDMAT}(m, s)$ unless m, s are small since it may be slow.

390 **10 How to find X Cheating a Little**

Say you want to find $f(213, 200)$. Since $\lceil \frac{2 \times 213}{200} \rceil = 3$ you could run $\text{BUDMAT}(213, 200)$. But the numbers are large! Following the line of reasoning in Section 7 we note that $d = 213 - 200 = 13$ and generalize the problem to finding an X such that

$$f(39k + 5 + 13, 39k + 5) \leq \frac{13k + X}{39k + 5}.$$

391 Lets look at the $k = 1$ case: $f(57, 44)$. Since $\lceil \frac{2 \times 57}{44} \rceil = 3$ you could run $\text{BUDMAT}(57, 44)$.
 392 But the numbers are small! Oh, thats a good thing! Lets say the answer is α . Run
 393 $\text{VLOWER}(57, 44, \alpha)$ to verify that its a lower bound. If it is then solve $\alpha = \frac{13+X}{39+5}$ to find
 394 X . The proof you did for $f(57, 44) \leq \frac{13+X}{39+5}$ can be modified to show $(\forall k \geq 1)[f(39k + 5 +$
 395 $13, 39k + 5) \leq \frac{13k+X}{39k+5}]$. In particular $f(213, 200) \leq \frac{13 \times 5 + X}{39 \times 5 + 5} = \beta$. Run $\text{VLOWER}(213, 300, \beta)$
 396 to verify the lower bound (if this does not work then the algorithm failed to find $f(57, 44)$).

397 This is cheating a little since we don't really know that the such an X exists. But it has
 398 so far. And we do verify in the end.

399 We leave it to the reader to generalize this procedure. We denote this algorithm
 400 $\text{CHEATALITTLE}(m, s)$.

401 **11 How to find X Cheating a Lot**

Say you want to find $f(1717, 1650)$. Since $\lceil \frac{2 \times 1717}{1650} \rceil = 3$ you could run $\text{BUDMAT}(1717, 1650)$. But the numbers are really large! Following the line of reasoning in Section 7 we note that $d = 1717 - 1650 = 67$ and generalize the problem to finding an X such that

$$f(201k + 42 + 67, 201k + 42) \leq \frac{67k + X}{201k + 42}.$$

402 Lets look at the $k = 1$ case: $f(310, 243)$. These numbers are still big!

403 Lets look at the $k = 0$ case: $f(109, 42)$. These numbers are small! Since $\lceil \frac{2 \times 109}{42} \rceil \geq 4$ you
 404 cannot run $\text{BUDMAT}(109, 42)$. But the situation is worse than that. Even if we bound
 405 $f(109, 42)$ the proof will not use BUDMAT and hence cannot be modified to get an upper
 406 bound for $f(201k + 42 + 67, 201k + 42)$. In fact, the answer for $f(109, 42)$ should have no
 407 bearing on our problem.

408 Except for one thing. Empirically it does. In all cases that we looked at the X obtained
 409 from knowing an upper bound on the $k = 0$ case of $f(3dk + a + d, 3dk + a)$ was the correct
 410 X for $k \geq 1$. We proceed as if this is always true.

411 We cannot use $\text{BUDMAT}(109, 42)$; however, there are other techniques that to find
 412 an upper bound on $f(m, s)$. They summarized in Section B. Use them. Lets say the

answer is α . Run $\text{VLOWER}(109, 42, \alpha)$ to verify that its a lower bound. If it is then solve
 $\alpha = \frac{X}{42}$ to find X . The proof you did for $f(109, 42) \leq \frac{X}{42}$ *cannot* be modified to show
 $(\forall k \geq 1)[f(201k + 42 + 67, 201k + 42) \leq \frac{67k+X}{201+42}]$. But you have a very good conjecture.
Run $\text{BUDMAT}(109, 42, \frac{67+X}{201+42})$. If it returns YES and a proof then modify the proof to
obtain $(\forall k \geq 1)[f(201k + 42 + 67, 201k + 42) \leq \frac{67k+X}{201+42}]$ (if this does not work then the
algorithm failed to find $f(1717, 1650)$). In particular $f(1717, 1658) \leq \frac{67 \times 5 + X}{201 \times 5 + 5} = \beta$. Run
 $\text{VLOWER}(1717, 1658, \beta)$ to verify the lower bound (if this does not work then the algorithm
failed to find $f(1717, 1650)$).

This is cheating a lot since we don't really know that the $k = 0$ case has any bearing on
the $k \geq 1$ case. But it has so far, and we verify in the end.

We leave it to the reader to generalize this procedure. We denote this algorithm
 $\text{CHEATALOT}(m, s)$.

12 A General Algorithm

We present an algorithm that we conjecture always finds $f(m, s)$ and operates in polynomial
time.

The reader should read Section B since we will be using FC, INT, and BUD which are
explained there. They are other methods to find or verify upper bounds on $f(m, s)$.

1. Input (m, s) .
2. If $m = s$ output 1. If $\gcd(m, s) = d \geq 1$ then call the algorithm recursively with
 $f(m/d, s/d)$. If $s = 2$ then output $\frac{1}{2}$. If $m < s$ then call the algorithm recursively to find
 $f(s, m)$ and output $\frac{m}{s} f(s, m)$.
3. Compute $\alpha = \text{FC}(m, s)$. Compute $\text{VLOWER}(m, s, \alpha)$ to see if α is a matching lower
bound. If it is then output α and stop.
4. Compute $\alpha = \text{INT}(m, s)$. Compute $\text{VLOWER}(m, s, \alpha)$ to see if α is a matching lower
bound. If it is then output α and stop.
5. If $\lceil \frac{2m}{s} \rceil = 3$ then:
 - a. Compute $\alpha = \text{CHEATALOT}(m, s)$. Compute $\text{VLOWER}(m, s, \alpha)$ to see if α is a
matching lower bound. If it is then output α and stop. (This might fail if the methods
of Section B do not work on the input they are given.)
 - b. Compute $\alpha = \text{CHEATALITTLE}(m, s)$. Compute $\text{VLOWER}(m, s, \alpha)$ to see if α is a
matching lower bound. If it is then output α and stop.
6. If $\lceil \frac{2m}{s} \rceil \geq 4$ then let $a = s$ and $d = m - a$. We seek $f(3d \times 0 + a + d, 3d \times 0 + a)$. Recursively
call $f(3d + a + d, 3d + a)$ (we could tell it to not bother with $\text{CHEATALOT}(m, s)$ since
that just asks to compute $f(a + d, a)$ using FC and INT). If the computation succeeds
and returns α then run $\text{BUD}(m, s, \alpha)$ to verify that $f(m, s) \leq \alpha$. If this is verified then
compute $\text{VLOWER}(m, s, \alpha)$ to see if α is a matching lower bound. If it is then output α
and stop.
7. If nothing above works then output **FAILED!**

This can be sped up by, upon first seeing m, s , see if any of the general theorems such as
those in Sections C and D apply to get an upper bound α and then run $\text{VLOWER}(m, s, \alpha)$.

13 Open Problems and Speculation

We would like to think that the algorithm in the last section will always work and hence
computing $f(m, s)$ is in P. But we've been down this road before where we think we can

compute all $f(m, s)$ only to come to a troublesome case which leads to a new technique and more co-authors. The following are possible outcomes: (1) we prove that the algorithm always works, (2) we keep running the algorithm and it always works but when the numbers get too big we can't tell, (3) we come across a value the algorithm does not work on and this leads to a new technique and more co-authors.

We believe that computing $f(m, s)$ is in P. One piece of evidence for this is that for all s , for all $m \geq s^3$, $f(m, s) = \text{FC}(m, s)$. Hence if you fix s then for large enough s the problem is *very easy*. One might call this Fixed Parameter *very tractable*.

We believe that $f(m, s)$ only depends on $\frac{m}{s}$. This seems provable.

A A Mixed Integer Program for $f(m, s)$

The following theorem shows that $f(m, s)$ always exists (as opposed to having better and better algorithms), is rational, and is computable. This theorem was independently discovered by Veit Elser, within the math-fun email list, in 2009.

► **Theorem 11.** *Let $m, s \geq 1$.*

1. *There is a mixed integer program with $O(ms)$ binary variables, $O(ms)$ real variables, $O(ms)$ constraints, and all coefficients integers of absolute value $\leq \max\{m, s\}$ such that, from the solution, one can extract $f(m, s)$ and a protocol that achieves this bound. This MIP can easily be obtained given m, s .*
2. *$f(m, s)$ is always rational. This follows from part 1.*
3. *In every optimal protocol for m muffins and s students all of the pieces are of rational size. This follows from part 1.*
4. *The problem of, given m, s , determine $f(m, s)$, is decidable. This follows from part 1.*

Proof. Consider the following (failed) attempt to solve the problem using linear programming.

1. The variables are x_{ij} where $1 \leq i \leq m$ and $1 \leq j \leq s$. The intent is that x_{ij} is the fraction of muffin i that student j gets.
2. For all $1 \leq i \leq m$, $1 \leq j \leq s$, $0 \leq x_{ij} \leq 1$.
3. For each $1 \leq i \leq m$, $\sum_{j=1}^s x_{ij} = 1$.
This says that the amount of muffin i that student 1 gets, students 2 gets, ..., student s gets all adds up to 1.
4. For each $1 \leq j \leq s$, $\sum_{i=1}^m x_{ij} = \frac{m}{s}$.
This says that the amount that student j gets from muffin 1, muffin 2, ..., muffin m all adds up to $\frac{m}{s}$.
5. For all $1 \leq i \leq m$, $1 \leq j \leq s$, $x_{ij} \geq z$.
6. Maximize z .

This does not work. The problem is that (say) x_{13} could be 0. In fact it is likely that some x_{ij} is 0. This makes $z = 0$. What we really want is

$$x_{ij} \neq 0 \implies x_{ij} \geq z$$

It is easy to show that $f(m, s) \geq \frac{1}{s}$. Hence every nonzero x_{ij} is $\geq \frac{1}{s}$. We will use this in our proof.

For $1 \leq i \leq m$, $1 \leq j \leq s$ modify the linear program above as follows.

1. Add variable y_{ij} which is in $\{0, 1\}$.
2. Add the constraint $x_{ij} + y_{ij} \leq 1$. Note that

- 497 ■ $x_{ij} = 0 \implies x_{ij} + y_{ij} \leq 1$, so the constraint imposes no condition on y_{ij} .
 - 498 ■ $x_{ij} > 0 \implies y_{ij} < 1 \implies y_{ij} = 0 \implies x_{ij} + y_{ij} = x_{ij}$.
 - 499 3. Add the constraint $x_{ij} + y_{ij} \geq \frac{1}{s}$. Note that
 - 500 ■ $x_{ij} = 0 \implies y_{ij} \geq \frac{1}{s} \implies y_{ij} = 1 \implies x_{ij} + y_{ij} = 1$
 - 501 ■ $x_{ij} > 0 \implies x_{ij} \geq \frac{1}{s}$ (since we know all non-zero pieces are $\geq \frac{1}{s}$) $\implies x_{ij} + y_{ij} \geq \frac{1}{s}$,
 - 502 so the constraint imposes no condition on y_{ij} .
 - 503 4. Replace the constraint $z \leq x_{ij}$ with $z \leq x_{ij} + y_{ij}$.
- If $x_{ij} = 0$ then the constraint

$$z \leq x_{ij} + y_{ij} = 1$$

is always met and hence is (as it should be) irrelevant. If $x_{ij} > 0$ then the constraint

$$z \leq x_{ij} + y_{ij} = x_{ij}$$

is the constraint we want.

Solve the resulting mixed integer program. Since all of the coefficients are rational the answer will be rational.

507

508 **B Other Methods**

We discuss three methods for finding an upper bound on $f(m, s)$.

The method from the following theorem is called *The Floor Ceiling Method* or just FC-method. Note that it is very fast and gives you the upper bound.

► **Theorem 12.** Assume that $m, s \in \mathbb{N}$ and $\frac{m}{s} \notin \mathbb{N}$.

$$f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.$$

Proof. Assume we have an optimal (m, s) protocol. Since $\frac{m}{s} \notin \mathbb{N}$ we can assume every muffin is cut into at least 2 pieces.

Case 1: Some muffin is cut into $u \geq 3$ pieces. Then some piece is $\leq \frac{1}{3}$.

Case 2: All muffins are cut into 2 pieces. Since there are $2m$ shares and s students both of the following happen:

- 517 ■ Some student gets $t \geq \lceil 2m/s \rceil$ shares, so some share is $\leq \frac{m}{s \lceil 2m/s \rceil}$.
- 518 ■ Some student gets $t \leq \lfloor 2m/s \rfloor$ shares, so some share x is $\geq \frac{m}{s \lfloor 2m/s \rfloor}$. $B(x) \leq 1 - \frac{m}{s \lfloor 2m/s \rfloor}$.

Putting together Cases 1 and 2 yields the theorem.

520

We denote the function from Theorem 12 $FC(m, s)$.

The other two methods are too long to describe fully here so we just sketch.

The *Interval Method* is a primitive version of the Buddy-Match method where we do not use symmetry and (since we have shares other than 2-shares and 3-shares) cannot use the Match in Buddy-Match. This method is fast and can be used to derive the answer. We denote the result $INT(m, s)$.

The *Buddy Method* is like the Buddy-Match Method only we do not use the Match part since we have shares other than 2-shares and 3-shares. And like the Buddy-Match Method this one is faster if you already have the answer. We denote the version where you do not already have an upper bound to check $BUD(m, s)$ and the version where you do $BUD(m, s, \alpha)$ where α is the bound.

532 **C** Everything You Ever Wanted to Know About $f(s+7, s)$

533 By either cheating a little (Section 10) or cheating a lot (Section 11) we have obtained
 534 formulas for $f(3dk+a+d, 3dk+a)$ for $1 \leq d \leq 50$ and $1 \leq a \leq 3d-1$ (a, d relatively primes).
 535 We present the results for $d=7$. Note that for most of the formulas the formula which is
 536 supposed to only hold for $k \geq 1$ also holds for $k=0$ (with a different proof).

537 **► Theorem 13.**

- 538 1. a. $f(8, 1) = 1$. For all $k \geq 1$, $f(21k+8, 21k+1) \leq \frac{7k+X}{21k+1}$ where $X = \frac{1}{2}$.
- 539 b. For all $k \geq 0$, $f(21k+9, 21k+2) \leq \frac{7k+X}{21k+2}$ where $X = 1$.
- 540 2. For all $k \geq 0$, $f(21k+10, 21k+3) \leq \frac{7k+X}{21k+3}$ where $X = \frac{4}{3}$.
- 541 3. For all $k \geq 0$, $f(21k+11, 21k+4) = \frac{7k+X}{21k+4}$ where $X = \frac{9}{5}$.
- 542 4. For all $k \geq 0$, $f(21k+12, 21k+5) \leq \frac{7k+X}{21k+5}$ where $X = 2$.
- 543 5. For all $k \geq 0$, $f(21k+13, 21k+6) \leq \frac{7k+X}{21k+6}$ where $X = \frac{13}{5}$.
- 544 6. For all $k \geq 0$, $f(21k+15, 21k+8) \leq \frac{7k+X}{21k+8}$ where $X = 3$.
- 545 7. For all $k \geq 0$, $f(21k+16, 21k+9) \leq \frac{7k+X}{21k+9}$ where $X = \frac{11}{3}$.
- 546 8. For all $k \geq 0$, $f(21k+17, 21k+10) \leq \frac{7k+X}{21k+10}$ where $X = 4$.
- 547 9. For all $k \geq 0$, $f(21k+18, 21k+11) \leq \frac{7k+X}{21k+11}$ where $X = \frac{9}{2}$.
- 548 10. For all $k \geq 0$, $f(21k+19, 21k+12) \leq \frac{7k+X}{21k+12}$ where $X = \frac{19}{4}$.
- 549 11. For all $k \geq 0$, $f(21k+20, 21k+13) \leq \frac{7k+X}{21k+13}$ where $X = 5$.
- 550 12. For all $k \geq 0$: $f(21k+22, 21k+15) = \frac{1}{3}$,
- 551 13. For all $k \geq 0$: $f(21k+23, 21k+16) = \frac{1}{3}$,
- 552 14. For all $k \geq 0$: $f(21k+24, 21k+17) = \frac{1}{3}$,
- 553 15. For all $k \geq 0$: $f(21k+25, 21k+18) = \frac{1}{3}$,
- 554 16. For all $k \geq 0$: $f(21k+26, 21k+19) = \frac{1}{3}$,
- 555 17. For all $k \geq 0$: $f(21k+27, 21k+20) = \frac{1}{3}$.

556 Note that the last few answers were $\frac{1}{3}$ and there is an equality. The $\frac{1}{3}$ follows from
 557 Theorem 14. The equality holds since we have proven that, for all $m > s$, $f(m, s) \geq \frac{1}{3}$.

558 **D** A Sample of General Theorems

559 In all cases a, d are relatively prime.

560 **► Theorem 14.** If $a \in \{2d+1, \dots, 3d-1\}$ then $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ where
 561 $X = \frac{a}{3}$, so $f(3dk+a+d, 3dk+a) \leq \frac{1}{3}$.

562 **► Theorem 15.** If $a \in \{1, \dots, 3d-1\}$, $a \neq d$, then $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ where
 563 $X = \max\{\frac{a}{3}, \frac{a+d}{5}, \frac{2a-d}{3}\}$.

564 **► Theorem 16.** If $1 \leq a \leq 3d-1$ and $5a \neq 7d$ then $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ where
 565 $X = \max\{\frac{a}{3}, \frac{a+d}{5}, \frac{a+2d}{6}, \frac{3a-2d}{4}\}$.

566 **► Theorem 17.** If $1 \leq a \leq \frac{5d}{7}$ and $a \neq \frac{2d}{3}$ then $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ where
 567 $X = \max\{\frac{2a}{5}, \frac{a+d}{6}\}$.

568 **► Theorem 18.** If $\frac{5d}{7} \leq a \leq d-1$ then $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ where
 569 $X = \max\{\frac{2a}{5}, \frac{3a-d}{4}\}$.

570 **► Theorem 19.** If $\frac{5d}{13} \leq a \leq \frac{13d}{29}$ and $a \neq \frac{2}{5}d$ then $f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}$ where
 571 $X = \max\{\frac{5a-d}{6}, \frac{a+d}{8}, \frac{3a}{7}\}$.

E If $m \geq s$ then $f(m, s) \geq 1/3$

Before showing the general technique we give an example.

Example: $f(19, 17) \geq \frac{1}{3}$.

We express $\frac{19}{17}$ as $\frac{57}{51}$ since other fractions will have a denominator of 51.

We initially divide the 19 muffins $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. There are now 57 pieces $\frac{1}{3}$ -shares. We initially give 11 students 3 $\frac{1}{3}$ -shares and 6 students 4 $\frac{1}{3}$ -shares. (In the proof below $W = 3$, $s_W = s_3 = 11$, and $s_{W+1} = s_4 = 6$.) A student who gets 3 (4) shares is called a *3-student* (*4-student*).

We describe a process whereby students give pieces of muffins, called gifts, to other students so that, in the end, all students have $\frac{57}{51}$. Each gift leads to a change in how the muffins are cut in the first place; however, there will never be a muffin of size $< \frac{1}{3}$.

Each 4-student has $\frac{4}{3} = \frac{68}{51}$ and hence has to give (perhaps in several increments) $\frac{68}{51} - \frac{57}{51} = \frac{11}{51}$ to get *down to* $\frac{57}{51}$. Realize that if a 4-student gives $\frac{11}{51}$ to a 3-student, then the 3-student now has $\frac{51}{51} + \frac{11}{51} = \frac{62}{51} > \frac{57}{51}$.

Each 3-student has $\frac{51}{51}$ and hence has to receive $\frac{57}{51} - \frac{51}{51} = \frac{6}{51}$ to get *up to* $\frac{57}{51}$.

Call the 11 3-students $g_1, \dots, g_{11}$.

Call the 6 4-students $f_1, \dots, f_6$.

We use a notation that we just give an example of:

f_1 gives x to g_1 by taking two $\frac{1}{3}$ -pieces, combining them, cutting off a piece of size x , giving it to g_1 while keeping the rest. g_1 takes the piece given to him and combines it with a $\frac{1}{3}$ piece. Notice that in terms of pieces we are taking three pieces of size $\frac{1}{3}$ (2 from f_1 and 1 from g_1) and turning them into 1 piece of size $\frac{2}{3} - x$ and one of size $\frac{1}{3} + x$. Hence we can easily rearrange how the muffins are cut.

$$x(f_1 \rightarrow g_1)$$

We need to make sure this procedure never results in a piece that is $< \frac{1}{3}$. In the above example (1) f_1 now has a piece of size $\frac{2}{3} - x$, hence we need $x \leq \frac{1}{3}$, (2) g_1 now has a piece of size $\frac{1}{3} + x$, which is clearly $\geq \frac{1}{3}$. Hence the only restriction is $x \leq \frac{1}{3}$.

1. $\frac{11}{51}(f_1 \rightarrow g_1)$. Now f_1 has $\frac{57}{51}$. YEAH. However, g_1 has $\frac{62}{51}$.
2. $\frac{5}{51}(g_1 \rightarrow g_2)$. Now g_1 has $\frac{62}{51} - \frac{5}{51} = \frac{57}{51}$. YEAH. However, g_2 has $\frac{51}{51} + \frac{5}{51} = \frac{56}{51}$.
3. $\frac{1}{51}(f_2 \rightarrow g_2)$. Now g_2 has $\frac{57}{51}$. YEAH. However, f_2 has $\frac{67}{51}$.
4. $\frac{10}{51}(f_2 \rightarrow g_3)$. Now f_2 has $\frac{57}{51}$. YEAH. However, g_3 has $\frac{61}{51}$.
5. $\frac{4}{51}(g_3 \rightarrow g_4)$. Now g_3 has $\frac{57}{51}$. YEAH. However, g_4 has $\frac{55}{51}$.
6. $\frac{2}{51}(f_3 \rightarrow g_4)$. Now g_4 has $\frac{57}{51}$. YEAH. However, f_3 has $\frac{66}{51}$.
7. $\frac{9}{51}(f_3 \rightarrow g_5)$. Now f_3 has $\frac{57}{51}$. YEAH. However, g_5 has $\frac{60}{51}$.
8. $\frac{3}{51}(g_5 \rightarrow g_6)$. Now g_5 has $\frac{57}{51}$. YEAH. However, g_6 has $\frac{54}{51}$.
9. $\frac{3}{51}(f_4 \rightarrow g_6)$. Now g_6 has $\frac{57}{51}$. YEAH. However, f_4 has $\frac{65}{51}$.
10. $\frac{8}{51}(f_4 \rightarrow g_7)$. Now f_4 has $\frac{57}{51}$. YEAH. However, g_7 has $\frac{59}{51}$.
11. $\frac{2}{51}(g_7 \rightarrow g_8)$. Now g_7 has $\frac{57}{51}$. YEAH. However, g_8 has $\frac{53}{51}$.
12. $\frac{4}{51}(f_5 \rightarrow g_8)$. Now g_8 has $\frac{57}{51}$. YEAH. However, f_5 has $\frac{64}{51}$.
13. $\frac{7}{51}(f_5 \rightarrow g_9)$. Now f_5 has $\frac{57}{51}$. YEAH. However, g_9 has $\frac{58}{51}$.
14. $\frac{1}{51}(g_9 \rightarrow g_{10})$. Now g_9 has $\frac{58}{51}$. YEAH. However, g_{10} has $\frac{52}{51}$.
15. $\frac{5}{51}(f_6 \rightarrow g_{10})$. Now g_{10} has $\frac{57}{51}$. YEAH. However, f_6 has $\frac{63}{51}$.
16. $\frac{6}{51}(f_6 \rightarrow g_{11})$. Now f_6 has $\frac{57}{51}$. YEAH. However, g_{11} has $\frac{57}{51}$. OH. thats a good thing!

YEAH- we are done.

Note that the first x was $\frac{11}{51} \leq \frac{1}{3}$ and the remaining x were all $\leq \frac{11}{51} \leq \frac{1}{3}$. Hence all pieces in the final protocol are $\geq \frac{1}{3}$.

End of Example

618 ► **Theorem 20.** For all $m \geq s$, $f(m, s) \geq \frac{1}{3}$.

619 **Proof.** Divide all the muffins into $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Initially distribute them as evenly as possible
 620 among the students. There will be a number W such that some students get W shares and
 621 some get $(W + 1)$ -shares. Let s_W (s_{W+1}) be the number of students who get W ($W + 1$)
 622 shares.

623 We do not need the following but are noting it anyway. If s does not divide $3m$ then
 624 $W = \frac{3m}{s}$ and s_W, s_{W+1} are unique and determined by:

$$\begin{aligned} 625 \quad Ws_W + (W + 1)s_{W+1} &= 3m \\ s_W + s_{W+1} &= s \end{aligned}$$

626 (Technically, if $s \mid 3m$ there are two possible values of W .)

627 A student who gets W ($W + 1$) shares we call a W -student ($(W + 1)$ -student). All
 628 W -students get $\frac{W}{3}$. All $(W + 1)$ -students get $\frac{W+1}{3}$.

629 A W -student must get $< \frac{m}{s}$: if a W -student got $> \frac{m}{s}$ then all students would get $> \frac{m}{s}$
 630 and hence there would be $> s\frac{m}{s} = m$ muffins total. A $(W + 1)$ -student must get $> \frac{m}{s}$: if
 631 a $(W + 1)$ -student got $< \frac{m}{s}$ then all students would get $< \frac{m}{s}$ and hence there would be
 632 $< s\frac{m}{s} = m$ muffins total.

633 Hence we have:

$$634 \quad \frac{m}{s} - \frac{W}{3} \leq \frac{1}{3} \tag{1}$$

$$635 \quad \frac{W+1}{3} - \frac{m}{s} \leq \frac{1}{3} \tag{2}$$

636
 637 Now we will need to smooth out the distribution so that everyone receives $\frac{m}{s}$. We will do
 638 this by doing a sequence of moves of the form $x(f_i \rightarrow g_j)$ or $x(g_i \rightarrow g_j)$. as defined in the
 639 example.

640 We will assume s_{W+1} and s_W are relatively prime (this only comes up in Claim 3 below).
 641 This is fine because if they have a common factor d , we can just use the procedure for the
 642 $\frac{s_{W+1}}{d}, \frac{s_W}{d}$ case repeated d times.

643 **Claim 1:**

- 644 1. If $s_{W+1} < s_W$ then $\frac{W+1}{3} - \frac{m}{s} > \frac{m}{s} - \frac{W}{3}$.
 645 2. If $s_W < s_{W+1}$ then $\frac{W+1}{3} - \frac{m}{s} > \frac{m}{s} - \frac{W}{3}$.

646 **Proof of Claim 1:**

$$\begin{aligned} s_{W+1} \times \frac{W+1}{3} + s_W \times \frac{W}{3} &= m \\ s_{W+1} \times \left(\frac{m}{s} + \frac{W+1}{3} - \frac{m}{s} \right) + s_W \left(\frac{m}{s} + \frac{W}{3} - \frac{m}{s} \right) &= m. \\ \left(s_{W+1} + s_W \right) \frac{m}{s} + s_{W+1} \left(\frac{W+1}{3} - \frac{m}{s} \right) + s_W \left(\frac{W}{3} - \frac{m}{s} \right) &= m \\ s \times \frac{m}{s} + s_{W+1} \left(\frac{W+1}{3} - \frac{m}{s} \right) + s_W \left(\frac{W}{3} - \frac{m}{s} \right) &= m \\ \frac{W+1}{3} - \frac{m}{s} &= \frac{s_W}{s_{W+1}} \left(\frac{m}{s} - \frac{W}{3} \right) \end{aligned}$$

Both parts follow.

End of Proof of Claim 1

We give the procedure to obtain $f(m, s) \leq \frac{1}{3}$. There are two cases.

Case 1: $s_{W+1} < s_W$. Hence by Claim 1 $\frac{W+1}{3} - \frac{m}{s} > \frac{m}{s} - \frac{W}{3}$.

Call the s_W W -students $g_1, \dots, g_{s_W}$.

Call the s_{W+1} $(W+1)$ -students $f_1, \dots, f_{s_{W+1}}$.

1. Let $x = \frac{W+1}{3} - \frac{m}{s}$. Note that $x \leq \frac{1}{3}$. Do $x(f_1 \rightarrow g_1)$. Now f_1 has $\frac{m}{s}$. YEAH. However, g_1 has $\frac{W}{3} + \frac{W+1}{3} - \frac{m}{s} > \frac{m}{s}$. (This is where we use $s_{W+1} < s_W$, or more accurately the consequence of that from Claim 1.)
2. Let $x = \frac{2W+1}{3} - 2\frac{m}{s}$. Do $x(g_1 \rightarrow g_2)$. Now g_1 has $\frac{m}{s}$. YEAH.
3. If g_2 has $> \frac{m}{s}$ then g_2 gives enough to g_3 so that g_2 has $\frac{m}{s}$. Keep up this chain of $g_1, g_2, g_3, \dots$ until there is a g_i such that g_i end up with $< \frac{m}{s}$ (though more than the $\frac{W}{3}$ that g_i had originally).
4. Do $x(f_2 \rightarrow g_i)$ where x is such that g_i will now have $\frac{m}{s}$.
5. Do $x(f_2 \rightarrow g_{i+1})$ where x is such that f_2 will now have $\frac{m}{s}$. Repeat the same chain of g_i 's as in step 3.
6. Repeat the above steps until you are done.

We need to show that (1) there is never a piece of size $< \frac{1}{3}$, and (2) the process ends with every student getting $\frac{m}{s}$.

Claim 2: The first gift is $\leq \frac{1}{3}$ and no gift is larger.

Proof of Claim 2: Let $C = \frac{W+1}{3} - \frac{m}{s}$ which is the size of the first gift. By equation (2) $C \leq \frac{1}{3}$.

Assume that all gifts so far have been $\leq C$. We analyze the three kinds of gifts and show that in all cases the gift is $\leq C$.

- $x(f_i \rightarrow g_j)$ where (1) initially f_i has $> \frac{m}{s}$, g_j has $< \frac{m}{s}$, and (2) after the gift f_i has $\frac{m}{s}$. When this occurs it is f_i 's first or second gift giving. (This happens in steps 1 and 5 above, and later as well.) Before the gift f_i has at least $\frac{m}{s}$ but at most $\frac{W+1}{3}$, so this gift has size at most $\frac{W+1}{3} - \frac{m}{s} = C$.
- $x(g_i \rightarrow g_{i+1})$ where (1) initially g_i has $> \frac{m}{s}$, g_j has $< \frac{m}{s}$, and (2) after the gift g_i has $\frac{m}{s}$. When this occurs g_i has received a gift once and this is g_i 's first time giving. (This happens in steps 2 and in the chain referred to in step 5.) Since g_i just received a gift of size $\leq C$ she has $\leq \frac{W}{3} + C$. Hence the gift is $\leq \frac{W}{3} - \frac{m}{s} + C \leq C$.
- $x(f_i \rightarrow g_j)$ where (1) initially f_i has $> \frac{m}{s}$, g_j has $< \frac{m}{s}$, and (2) after the gift g_j has $\frac{m}{s}$. This will be f_i 's first time giving. (This happens in step 4 above.) Before the gift f_i has at least $\frac{W}{3}$ but at most $\frac{m}{s}$, so this gift has size at most $\frac{m}{s} - \frac{W}{3} \leq C$ (by Claim 1).

Claim 3: If s_W and s_{W+1} are relatively prime then the process terminates with all students having $\frac{m}{s}$.

Proof of Claim 3:

In each step all of the f_i have at least $\frac{m}{s}$. In each step the number of students who have the correct amount of muffin goes up. One may be worried that at some point we will try to do step 4 (for example) of the procedure and there will be no g_i left who need more muffin. But this is not possible because until the process terminates the f 's always have more muffin than they need, so there is always a g with insufficient muffin.

One may also be worried that eventually we will get all of the f 's to have $\frac{m}{s}$, but the g 's will not all have $\frac{m}{s}$. This is not possible either, because whenever we only make gifts from f to g when there is no g with more than $\frac{m}{s}$.

693 Finally, if s_W and s_{W+1} are not relatively prime, it is possible that the procedure will
 694 terminate early because in step 5 the size of the donation x is 0. If this occurred it would
 695 mean that there is some subset of F f 's and G g 's each of which having exactly $\frac{m}{s}$, who only
 696 made donations amongst themselves. But then $\frac{F}{G} = \frac{s_{W+1}}{s_W}$, a contradiction.

697 **End of Proof of Claim 3**

698 **Case 2:** $s_W < s_{W+1}$. This is similar to Case 1 except that instead of f_1 giving g_1 so that f_1
 699 has $\frac{m}{s}$, f_1 gives to g_1 so that g_1 has $\frac{m}{s}$. Hence we have a chain of f_i 's instead of a chain of
 700 g_i 's. ◀

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