A FIVE-COLOR THEOREM FOR GRAPHS ON SURFACES

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ABSTRACT. We prove that if a graph embeds on a surface with all edges suitably short, then the vertices of the graph can be five-colored. The motivation is that a graph embedded with short edges is locally a planar graph and hence should not require many more than four colors.

Introduction. It is well known [4, 15] that a graph embedded on a surface of genus \( k > 0 \), the sphere with \( k \) handles, can always be \( H(k) \)-colored, where \( H(k) = \left\lfloor \frac{7 + \sqrt{48k + 1}}{2} \right\rfloor \); \( H(k) \) is called the Heawood number of the surface. A variety of properties are known which ensure that an embedded graph needs significantly fewer than \( H(k) \) colors, for example, large girth [12, 13], few triangles (for graphs on the sphere or torus and, more generally, on surfaces of nonnegative Euler characteristic) [9, 11] and Eulerian properties of the (topological) dual graph [10].

On the other hand, one can look for properties which ensure that an embedded graph is locally a planar graph and hence needs not many more than four colors. In this spirit, Mycielski [14] has asked whether for every surface \( S \) there is an \( E > 0 \) such that a graph embedded on \( S \) with edges of length less than \( E \) can be five-colored. We restate Mycielski’s question in terms of an explicit metric and then answer it in the affirmative for all surfaces. Work of Albertson and Stromquist [3] has already settled the case for the torus \( (k = 1) \), and we use many of their techniques in our proof. Also in [3] examples due to J. P. Ballantine and S. Fisk are given which show that no similar result for four-colorability is possible for any surface of positive genus.

A surface of genus \( k \geq 1 \) can be represented as a \( 4k \)-sided polygon with pairs of sides identified [7, 16, 18]. If a graph is embedded on a surface of genus \( k \geq 1 \), we obtain a representation \( G_k \) of \( G \) in and on the boundary of the \( 4k \)-gon. Without loss of generality we take the polygon to be a regular \( 4k \)-gon with sides of unit length; we call this the standard \( 4k \)-gon, \( P_k \). Each edge of \( G \) is represented in \( G_k \) by one or more arcs in \( P_k \) (if an edge crosses the boundary of \( P_k \), it is divided into pieces). As explained in [5, p. 16], we may assume that each arc of \( G_k \) is a polygonal arc; then by the length of an edge of an embedded graph \( G \) we mean the sum of the lengths of its polygonal arcs in the representation \( G_k \). Thus length is always defined in terms of a fixed representation of the graph on a standard polygon.

Our main result is the following.
Theorem 1. Suppose $G$ has a 2-cell embedding on a surface of genus $k \geq 1$ and suppose $G$ has a representation $G_k$ on the standard $4k$-gon such that every edge of $G$ has length less than $\varepsilon = 1/5$. Then $G$ can be five-colored.

For the torus Albertson and Stromquist [3] have shown that a graph embedded with all noncontractible cycles of length at least 8 can be five-colored. This implies Theorem 1 with $k = 1$ and $\varepsilon = 1/7$ since all noncontractible cycles on $P_k$ have (Euclidean) length at least 1. Stromquist [17] has more recently shown that 5-colorability follows for toroidal graphs provided all noncontractible cycles have length at least 4, giving Theorem 1 with $k = 1$ and $\varepsilon = 1/3$. They conjecture that for each $k \geq 1$, there is a bound $b_k$ such that every graph embedded on the surface of genus $k$ with all noncontractible cycles of length at least $b_k$ can be 5-colored. The value of the bound must depend on $k$ since there are 6-chromatic graphs of arbitrarily large girth [6]. A proof of their conjecture would give Theorem 1 with $\varepsilon = 1/(b_k - 1)$; however, our result holds with a fixed value of $\varepsilon$ for all surfaces. On the other hand, their results and conjecture are more natural in that they use a metric intrinsic to the graph whereas Theorem 1 relies upon an external geometric one.

Although short edges (as defined here) imply that all noncontractible cycles are long, the converse does not hold; for all surfaces there are graphs with all noncontractible cycles long and with some long edges in every representation $G_k$. For example, the graph on the double torus in Figure 1 has all noncontractible cycles of length at least 6; it is a 4-colorable graph.

There is no loss of generality in our interpretation of Mycielski's question and in our definition of edge length for the following reasons. Suppose $S_k$, the sphere with
Suppose we define “nice” to mean that $S_k$ is a differentiable manifold ([16, §§2–3] gives a good introduction to this subject), and suppose we assume all edges of $G$ are piecewise differentiable curves on $S_k$ (as shown in [5], we lose no generality in this assumption). Then the length of each edge of $G$ can be determined by an integral; we denote the length of a piecewise differentiable curve $\gamma$ on $S_k$ by $\|\gamma\|_1$. Furthermore, there is a homeomorphism $f$ from $S_k$ to the standard $4k$-gon $P_k$ in the plane, which is also differentiable. Then $f$ will map edges (or arbitrary piecewise differentiable curves $\gamma$) to piecewise differentiable curves in $P_k$; we denote the resulting lengths by $\|f(\gamma)\|_2$. Since $S_k$ and $P_k$ are compact, there are constants $c_1$ and $c_2$ such that $c_1\|\gamma\|_1 \leq \|f(\gamma)\|_2 \leq c_2\|\gamma\|_1$ for all piecewise differentiable curves $\gamma$ on $S_k$. Then we have the following consequence of Theorem 1.

**Corollary 1.** Suppose $G$ has a 2-cell embedding on a differentiable manifold of genus $k \geq 1$ and suppose every edge of $G$ is piecewise differentiable. Then if every edge of $G$ has length less than $1/(5c_2)$, $G$ can be five-colored.

This follows by noting that the proof of Theorem 1 holds as well when the edges in the representation $G_k$ are piecewise differentiable.

**Background in topological graph theory.** We use basic graph theory terms as found in [18]. We consider only simple graphs and their 2-cell embeddings on surfaces, i.e. embeddings in which the interior of every face is a contractible (or null-homotopic) region. A 2-cell embedding implies that the graph is connected; there is no loss of generality in considering only 2-cell embeddings since any embedding of a connected graph can be transformed into a 2-cell embedding by suitably cutting handles of the surface without affecting the graph embedding (see [18, p. 54]).

A cycle in a graph embedded on a surface is said to be contractible or noncontractible according as it is or is not homotopic to a point on the surface; we abbreviate the latter by calling it an nc-cycle. A cycle in an embedded graph is said to be null-homologous or non-null-homologous if it is an nc-cycle whose removal does or does not, respectively, disconnect the graph; we abbreviate the latter by calling it an nnh-cycle. In Figure 1 the graph shown on the double torus contains $C_1 = \{1, 9, 14\}$, a contractible cycle, and $C_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ or $C_3 = \{1, 10, 11, 12, 13, 14\}$, a non-null-homologous cycle. In Figure 2 a noncontractible and null-homologous cycle $C$ is marked in dashed lines; such cycles are present in the graph of Figure 1 but are long and not as illustrative. For the torus only, all nc-cycles are nnh, but for other surfaces the distinction is important (e.g. see [1]).

A cycle in a graph is said to be minimal if it contains no diagonal. If $C$ is a minimal nc-cycle, we arbitrarily give $C$ an orientation and define $R(C)$ and $L(C)$ to be the set of neighbors of $C$ which lie, respectively, to the right and to the left of $C$ on the surface as $C$ is traversed following the given orientation. $R(C)$ and $L(C)$ need not be disjoint.
The final standard topological fact which we shall use is the following. Suppose \( G \) has a 2-cell embedding on \( S_k, k \geq 1 \), and contains \( k \) disjoint, pairwise nonhomotopic, \( n_n \)-cycles \( C_1, C_2, \ldots, C_k \). Then deleting the vertices of \( C_1, C_2, \ldots, C_k \) and their incident edges leaves a planar graph, since the elimination of \( C_1, \ldots, C_k \) can be performed by cutting the surface along these cycles and sewing in \( 2k \) discs, leaving a sphere [7, p. 63]. Further, if \( C_1, \ldots, C_k \) are also minimal, we define \( G(C_1, C_2, \ldots, C_k) \) to be the planar graph obtained by adding \( 2k \) vertices to \( G - \{ \ldots, C_k \} \), two for each \( i = 1, \ldots, k \): let \( x'_R \) be adjacent to each vertex of \( R(C_i) \) and \( x'_L \) adjacent to each vertex of \( L(C_i) \).

The next three lemmas are crucial to the proof of Theorem 1; the proofs of the first two can be found in [3]. Although in [3] these results are stated only for the torus, they were designed to be valid for all surfaces and hence yield Lemmas 1 and 2 as stated. Let a graph \( G \) have a 2-cell embedding on a surface of genus \( k \geq 1 \). The embedding is said to be orderly if \( G \) is a triangulation, if every contractible 3-cycle is a face boundary, and if every contractible 4-cycle is either the first neighbor cycle of a vertex of degree 4 or the modulo 2 sum of two face boundaries with an edge in common.

**Lemma 1.** Let \( G \) be a triangulation of a surface and \( G_0 \) the orderly triangulation obtained by deleting all vertices interior to a contractible 3- or 4-cycle and by subdividing any resulting quadrilateral. If \( G_0 \) can be 5-colored, then so can \( G \).

**Lemma 2.** Suppose \( G \) has an orderly embedding on a surface and let \( C \) be a minimal \( n_c \)-cycle of length at least 4. Then within the induced (and embedded) subgraph of
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$C \cup R(C)$, there is a minimal $nc$-cycle which either has even length or contains a vertex of degree 4 (in $G$).

Such a cycle is called nice.

Suppose $C_1$ and $C_2$ are two disjoint, nonhomotopic, $nnh$-cycles in an embedded graph. We define $d(C_1, C_2)$ to be the length (number of edges) of the shortest path from a vertex of $C_1$ to one of $C_2$. We define $d(C_i, C_j)$, $i = 1, 2$, to be the length of the shortest path joining two vertices of $C_i$, say $v$ and $w$ where possibly $v = w$, such that the path plus one segment of $C_i$ joining $v$ and $w$ is an $nc$-cycle, not homotopic to $C_i$. The resulting shortest $nc$-cycle is called $C_i^*$; the idea is that $C_i^*$ is the shortest cycle going around the same handle as $C_i$, but in a “different” direction. (For more details see [2, 3].)

We sketch the proof of the next result. It is only slightly different than that presented in [3], but it illustrates the coloring techniques involved.

**Lemma 3.** Suppose $G$ is an orderly triangulation of a surface of genus $k \geq 1$ and contains $k$ nice, $nnh$-cycles $C_1, \ldots, C_k$ which are pairwise disjoint and nonhomotopic. If $d(C_i, C_j) \geq 4$ for all $i, j \in \{1, \ldots, k\}$, then $G$ can be 5-colored.

**Proof.** Form $G(C_1, \ldots, C_k)$ as defined above. Note that this graph is a triangulation of the sphere and that the induced subgraph on each set $R(C_i)$ and $L(C_i)$ is a cycle. Then $G(C_1, \ldots, C_k)$ can be 4-colored [4] from which $G - \{C_1, \ldots, C_k\}$ inherits a 4-coloring which we shall extend to a 5-coloring of $G$. Each cycle in $\{R(C_i), L(C_i): i = 1, \ldots, k\}$ has been colored with (at most) 3 colors (since, for example, all vertices of $R(C_i)$ are adjacent to $x_k^i$). Suppose $L(C_i)$ and $R(C_i)$ have received the same triple of colors, say $\{1, 2, 3\}$. Then colors $\{4, 5\}$ can be alternated on $C_i$; if $C_i$ has odd length, alternate these colors, leaving the vertex of degree 4 to the end at which point it can receive one of the 5 colors. If $L(C_i)$ and $R(C_i)$ have different triples, say $\{1, 2, 3\}$ and $\{1, 2, 4\}$, we replace colors $\{3, 4\}$ by color 5 on $L(C_i)$ and $R(C_i)$, and use colors $\{3, 4\}$ on $C_i$ as above. Since $d(C_i, C_j) \geq 4$, no two vertices from distinct cycles in $\{L(C_i), R(C_i): i = 1, \ldots, k\}$ are adjacent; thus this 5-coloring is proper.

**Main results.** We now prove our main result. We need to consider $G$ embedded on a surface and simultaneously its representation $G_k$ on $P_k$; when we alter $G$ or $G_k$ we carry out the corresponding alteration on the other.

**Proof of Theorem 1.** We assume $G$ has a 2-cell embedding on a surface of genus $k \geq 1$ and a representation $G_k$ on $P_k$ with all edges of length less than $\epsilon = 1/5$. As in [3] we begin by extending $G$ to an orderly triangulation of the surface.

First we subdivide every nontriangular face by adding a vertex adjacent to all vertices on the face boundary (and add these new vertices and edges to $G_k$). If any new edge has length $\epsilon$ or more, we subdivide it by adding new vertices along the edge. We repeat the above process until the resulting graph $G'$ is a triangulation with all edges of length less than $\epsilon$. Finally we create $G''$ by erasing all vertices inside a contractible 3- or 4-cycle and subdividing any resulting quadrilaterals. If $G''$ can be
5-colored, then so can $G'$ by Lemma 1. Then $G$ inherits a 5-coloring from $G'$ since no edge of $G$ was subdivided.

Thus we assume $G$ is an orderly triangulation and try to 5-color it. For each $i = 1, \ldots, k$ we consider the “handle” in the polygon $P_k$ with sides labelled $a_i$, $b_i$, $a_i^{-1}$, and $b_i^{-1}$ (see Figure 2); note that one point $S$ is common to all $4k$ sides of $P_k$. Let $p_i$ be the set of all points of $P_k$ at distance 1/2 from the side $b_i$ (see Figure 2). Thus $p_i$ is a path from the midpoint of $a_i$ to the midpoint of $a_i^{-1}$ and represents an nnh-cycle on the original surface (but which is not necessarily a cycle in the graph).

Let $L_i$ be the set of all points of $P_k$ which lie to the left of $p_i$, as it is traversed from $a_i$ to $a_i^{-1}$, and within distance $\epsilon$ of $p_i$.

We claim that within $L_i$ there is a path in $G_k$, starting and ending at corresponding edges or vertices of $a_i$ and $a_i^{-1}$, which represents an nnh-cycle in $G$. To find such a path, color a region (or face) of $G_k$ blue if it meets the set $L_i$ but does not cross $p_i$. Since all edges have length less than $\epsilon$ and $G$ is a 2-cell embedding, one component of the boundary of the blue region lies in $L_i$, giving the path in $G_k$ and the corresponding nnh-cycle in $G$. Within the latter cycle, find $C'_i$ which is a minimal nc-cycle.

Let $R(C'_i)$ be the set of neighbors of $C'_i$ which lie to the right of $C'_i$, as it is traversed from $a_i$ to $a_i^{-1}$ in $G_k$. By Lemma 2 we can find a nice nnh-cycle $C_i$ within $C'_i \cup R(C'_i)$, all vertices and edges of which lie within $\epsilon$ of $p_i$ in $P_k$.

Clearly $C_1, C_2, \ldots, C_k$ are pairwise disjoint and nonhomotopic. By Lemma 3, $G$ is 5-colorable provided $d(C_i, C_j) > 3$ for all $i, j \in \{1, 2, \ldots, k\}$. The shortest path from $C_i$ to $C_j$ (if $i \neq j$) and the shortest path from $C_i$ to $C_j$ which induces an nc-cycle $C^*_i$ lie along a path (in $P_k$) from $C_i$ to $S$ and from (another copy of) $S$ to $C_j$ (or $C_j$). Such a path has (Euclidean) length at least $2(1/2 - \epsilon)$. Thus $d(C_i, C_j) > (1 - 2\epsilon)/\epsilon \geq 3$ when $\epsilon \leq 1/5$.

We can more easily see that short edges ensure 7-colorability.

**Theorem 2.** Suppose $G$ has a 2-cell embedding on a surface of genus $k \geq 1$ and a representation $G_k$ on $P_k$ such that every edge of $G$ has length less than $\epsilon = 1/2$. Then $G$ can be 7-colored.

**Proof.** As in the proof of Theorem 1 we may alter $G$ to become a triangulation with all edges of length less than $\epsilon$; we do not require the graph to be orderly and so do not concern ourselves with separating 3- and 4-cycles. As before we find minimal nnh-cycles $C'_1, \ldots, C'_k$ which are pairwise disjoint and nonhomotopic; these cycles need not be nice. The shortest path from $C'_i$ to $C'_j$ has (Euclidean) length at least $1/2 - \epsilon + 1/2$. Thus $d(C'_i, C'_j) > (1 - \epsilon)/\epsilon \geq 1$, when $\epsilon \leq 1/2$, and no vertex of $C'_i$ is adjacent to one of $C'_j$. Removing the cycles $C'_1, \ldots, C'_k$ leaves a planar graph which can be 4-colored; at most 3 more colors are needed on the cycles $C'_i$, and no coloring conflicts occur in this 7-coloring.

The contrapositive of Theorems 1 and 2 is worth noting.

**Corollary 2.** Let $G$ be a 6- (respectively 8-) chromatic graph. Then in every embedding of $G$ on a surface of genus $k \geq 1$ (2) there are edges of length at least $1/5$ ($1/2$).
Presumably there are $k$-colorability results for $k = 6$ and $k \geq 8$ similar to those of Theorems 1 and 2. It is a bit surprising that the $e$ of these results does not depend upon $k$; however, if we had chosen our $4k$-gon to be regular with sides of length $s(k)$, as for example with a regular polygon with unit radius or unit area, then the same proofs would show that a graph with all edges of length less than $e = s(k)/5$ (or $e = s(k)/2$) can be 5-colored (7-colored).

We note that Theorems 1 and 2 can be interpreted to read that a “locally planar” graph embedded on a surface needs “few” colors. Albertson and Stromquist have called an embedded graph *locally planar* if there is an $i > 1$ such that the $i$th neighborhood of every vertex $v$ (i.e. the induced subgraph on $v$ and all vertices at distance at most $i$ from $v$) is embedded in a subset of the surface homeomorphic to a subset of the plane. Graphs which satisfy the hypotheses of Theorems 1 or 2 are locally planar since the second (first) neighborhood of each vertex lies in the representation $G_k$ with a circle of radius $2/5$ ($1/2$) and in $P_k$ each noncontractible cycle has (Euclidean) length at least one.

We conclude with two questions. Although the qualitative nature of Theorem 1 may be its main importance, it would be nice to know or to bound the constant $c_2$ of Corollary 1. In particular, if the embedding surface is taken to be one with all nc-cycles (of the surface) of length at least one, is there an edge length bound in terms of this unit of measure?

We ask a question which is a variant on one in [3]. In the proof of Theorem 1 (and of Lemma 3) the fifth color is used on relatively few vertices, about half of those of the $C_i$'s or of $R(C_i) \cup L(C_i)$. In [8] it is shown that by removing at most $O((\log k)/\sqrt{k})$ vertices of a graph embedded on a surface of genus $k$ with $n$ vertices, a planar graph results. Hence all but $O((\log k)/\sqrt{k})$ vertices can be 4-colored. Are there constants $M(k)$ such that a graph embedded on a surface of genus $k \geq 1$ with all edges suitably short can have all but $M(k)$ vertices 4-colored?

ADDENDUM. These same techniques can be applied to nonorientable surfaces to show that graphs embedded on these surfaces with (similarly) short edges also can be five-colored.

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