

Some Hat Problems and Their Answers And Some Points

Exposition by William Gasarch

1) N people stand in a line numbered $1, 2, 3, \dots, N$. If $i < j$ then person i can see person j 's hat color. Hats (Red and Blue) are put on heads randomly. People, in order $1, 2, 3, \dots, N$ will try to guess their hat color OR pass. Maximize the probability that there will be at least 1 real answer and all real answers are correct.

For $i = 1$ to N :

Person i does the following: If nobody has said RED yet and all the hats ahead of him are BLUE then he says RED. Otherwise he passes. (Note that person 1 will say RED if nobody before him has said RED.)

When will this fail: If ALL of the hats are BLUE then this will fail.

BUT this will work in all other cases. Assume the sequence is

$$\{R, B\}^{n-L-1}RB^L$$

for some $L \geq 0$. The first $n - L - 1$ will all pass since they see the R . The R will say R and be right.

So the scheme fails on B^n but succeeds otherwise. So prob of success is $1 - \frac{1}{2^n}$.

(I think this is optimal.)

POINT: I originally didn't have one for this problem but one of the commenters misread the problem (or perhaps I miswrote it) and thought that there was an adversary but the players could randomize. So a POINT here is that there are two kinds of randomization. Frankly I find his problem more interesting since you are trying to outwit an adversary. Here is a sketch of the solution based on what I have above. Above the 'bad' string is all B's. In his solution the players PICK a bad string r at random and do what I do above but with that bad string instead of the B's.

2) Same as problem 1 but the number of people is infinite and we need to have the people right all

but a finite number of times.

If $x, y \in \{R, B\}^\omega$ then we say $x \equiv y$ if they differ only finitely often. \equiv is an equivalence relation and hence a partition. Ahead of time the players all agree on representatives of each equivalence class.

Player i sees all but a finite number of people so he can determine equiv class they are in. He then guesses the color of i in the agreed upon representative.

The actual coloring differs finitely much from the representative. Hence the actual coloring differs only finitely much from what is said. Hence only a finite number are wrong.

POINT: Many students don't like this one since the people have to have uncountable memory in the head. Doesn't bother me, not even a little. One blogger said that it makes him doubt AC (Banach-Tarski bothers me a lot more). There is an article that proves AC is needed. That is, there is a model of ZF without choice where the theorem is false.

2) Same as problem 2 but only allowed ONE error.

I'll do the POINT first this time. I was sure this could not be done but never bothered writing down a proof. In fact, I just know that any bound on the errors was not possible. If this was a TV show I would say its time for me to eat humble pie- YES it can be done. (I've only heard the expressions "eat humble pie" and "eat crow" on TV shows, never in real life.)

Person 1 sees the truth (except for his own hat) and hence knows how much it differs from the representative. Person 1 says RED if it differs from representative an even number of times and Blue if odd. The rest of the players can then determine if they agree or not with the representative.

One can show you can't do better than just one wrong.

SECOND POINT: There is a version of problems 2 and 3 where everyone sees all hats but their own and they all shout the answer at the same time. For this one the answer to 2 is the same. For 3 I am SURE it can't be done. I might be looking at a second helping of humble pie.

3) n people, c hat colors, everyone sees everyone, they all say simultaneously the color, want to

maximize how many get it right. Adversary places the hats. Want upper and lower bound.

The people are $0, 1, 2, \dots, n - 1$. The hats are $0, 1, 2, \dots, c - 1$.

Person i acts as though he thinks the SUM mod c of the hats is i . So he computes the sum of what he sees which we call S_i . He then guesses x such that $x + S_i \equiv i \pmod{c}$.

Assume the actual sum mod c is S . Then all people who are $\equiv S \pmod{c}$ are correct. That's $\lfloor n/c \rfloor$.

Can you do better? No! Fix some protocol. Then assign hats RANDOMLY. For any particular person the probability of guessing correctly is $\frac{1}{c}$. Hence the expected number of correct guesses is $\frac{n}{c}$. Hence there IS some way to place hats so its $\lfloor n \rfloor c$.

POINT: Hat puzzles are fun and people like them. Hence its GREAT that they can also be used as an intro to the prob method, or at least early on in learning the prob method.