Some Hat Problems and Their Answers and Some Points

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For all of the problems we discuss

- There are a number of people, either some positive $n \in \mathsf{N}$ or countable.
- The number of hat colors is 2. It is an implicit exercise to see what happens with $c \in \mathsf{N}$ hat colors, or even a countable number of hat colors.
- The people will have hats put on their heads by an adversary.
- Every person can see every hat but their own.
- The people will all shout out a guess for their hat color at the same time. Their goal is to minimize how many of the guesses are wrong.
- The people can meet ahead of time to discuss and agree upon a strategy. We call this the *People-Strategy*. The People-Strategy must be deterministic.
- The adversary knows what the People-Strategy is. The adversary uses this to devise a way to maximize the number of guesses they get wrong. We call this the *Adversary-Strategy*. Note that the Adversary-Strategy can depend on the People-Strategy.

We view this more simply as a game:

- The adversary puts hats on everyone's head.
- The people all guess at the same time what their hat color is.
- The people's (adversary's) goal is to minimize (maximize) how many people guess wrong.

1 Finite Number of People

Theorem 1.1 *Assume the game is played with n people where n is even.*

1. *There is a People-Strategy where at least $n/2$ of them get their hat color right.*

2. *There is an Adversary-Strategy where at most $n/2$ of them get their hat color right.*

Proof:

1) Here is the People-Strategy: Partition the people into $n/2$ sets of 2 each. For each set we give a strategy where are least one guesses their hat color. We call the two people Alice and Bob.

Alice guesses Bob's hat color.

Bob guesses the opposite of Alice's hat color.

We leave it to the reader to prove that at least one will get their hat color correct.

2) Fix a strategy for the people.

Consider what happens if the adversary places hats on people's heads *at random*. We do not claim this is the Adversary-Strategy; however, it will be useful to ask what happens if he did this.

Look at one person Alice. It is easy to see that the probability that Alice's strategy gives the right answer is $\frac{1}{2}$. Hence the expected number of right answers is $\frac{n}{2}$. Therefore there is SOME way of putting hats on people so that the number of correct guesses is $\leq \frac{n}{2}$. That way is the Adversary-Strategy. ▮

2 Countable Number of People

Theorem 2.1 *Assume the game is played with a countable number of people.*

1. *There is a People-Strategy where only a finite number of guesses are wrong.*

2. *For all $w \in \mathsf{N}$ there is an Adversary-Strategy where at least w are wrong.*

Proof:

1) Here is the People-Strategy:

We represent a way that hats are put on people by an infinite string of R's and B's. The intent is that person i has hat color the ith element of the string. The set of all such strings is $\{R, B\}^{\omega}$.

If $x, y \in \{R, B\}^{\omega}$ then we say $x \equiv y$ if they differ only finitely often. $\equiv$ is an equivalence relation and hence gives a partition. Ahead of time, the players all agree on a representative of each equivalence class.

Player i sees all but his own hat so he can determine the equivalence class they are in. He then guesses the color of i in the agreed upon representative.

The actual coloring differs finitely often from the representative. Hence the actual coloring differs only finitely often from what the guesses. Hence only a finite number are wrong.

2) Assume, by way of contradiction, that there is a People-Strategy P and a $i \in \mathsf{N}$ such that no matter how the hats are placed on the people's heads, i will be incorrect.

Let n be an even number to be chosen later. We give a People-Strategy for the n-person problem that has $< \frac{n}{2}$ errors, contradicting Theorem 1.1.

The People-Strategy: People $1, 2, \ldots, n$ all shout what they would shout if they were doing strategy P and $n+1, n+2, \ldots$ were all RED. By our assumption at most w of them make a wrong guess. Take $n = 2i + 2$. The strategy guarantees at most $i = \frac{n}{2} - 1$ are wrong, contradicting Theorem 1.1. $\blacksquare$