The Kruskal Tree Theorem
Exposition by William Gasarch

1 Introduction

In 1960 Joe Kruskal [1] proved that the set of trees under the minor ordering is a well quasi order (wqo). (We will define wqo later.) In 1963 Nash-Williams [2] provided a simpler proof. This document is an exposition of the proof of Nash-Williams.

2 Well Quasi Orders

Definition 2.1 A set together with an ordering $(X, \preceq)$ is a *well quasi ordering* (wqo) if for any sequence $x_1, x_2, \ldots$ there exists i, j such that $i < j$ and $x_i \preceq x_j$. We call this i, j an *uptick*

Note 2.2 If $(X, \preceq)$ is a wqo then its both well founded and has no infinite antichains.

Lemma 2.3 *Let $(X, \preceq)$ be a wqo. For any sequence $x_1, x_2, \ldots$ there exists an infinite ascending subsequence.*

Proof: Let $x_1, x_2, \ldots,$ be an infinite sequence. Define the following coloring:
$COL(i, j) =$

- UP if $x_i \preceq x_j$.
- DOWN if $x_j \prec x_j$.
- INC if x_i and x_j are incomparable.

By Ramsey's theorem there is either an infinite homog UP-set, an infinite homog DOWN-set or an infinite homog INC-set. We show the last two cannot occur.

If there is an infinite homog DOWN-set then take that infinite subsequence. That subsequence violates the definition of well quasi ordering.

If there is an infinite homog INC-set then take that infinite subsequence. That subsequence violates the definition of well quasi ordering. ∎

We now redefine wqo.

Definition 2.4 A set together with an ordering $(X, \preceq)$. is a *well quasi ordering* (wqo) if one of the following equivalent conditions holds.

- For any sequence $x_1, x_2, \ldots$ there exists i, j such that $i < j$ and $x_i \preceq x_j$.
- For any sequence $x_1, x_2, \ldots$ there exists an *infinite* ascending subsequence.

3 If X and Y are wqo then $X \times Y$ is wqo

Definition 3.1 If $(X, \preceq_1)$ and $(Y, \preceq_2)$ are wqo then we define $\preceq$ on $X \times Y$ as $(x,y) \preceq (x',y')$ if $x \preceq_1 y$ and $x' \preceq_2 y'$.

Lemma 3.2 *If $(X, \preceq_1)$ and $(Y, \preceq_2)$ are wqo then $(X \times Y, \preceq)$ is a wqo ($\preceq$ defined as in the above definition).*

Proof: Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ be an infinite sequence of elements from $A \times B$.

Define the following coloring:
$COL(i,j) =$

- UP-UP if $x_i \preceq x_j$ and $y_i \preceq y_j$.
- UP-DOWN if $x_i \preceq x_j$ and $y_j \preceq y_i$.
- UP-INC if $x_i \preceq x_j$ and y_j, y_i are incomparable.
- DOWN-UP, DOWN-DOWN, DOWN-INC, INC-UP, INC-DOWN, INC-INC are defined similarly.

By Ramsey's theorem there is a homog set in one of those colors. If the color has a DOWN in it then there is an infinite descending sequence within either $x_1, x_2, \ldots$, or $y_1, y_2, \ldots$ which violates either X or Y being a wqo. If the color has an INC in it then there is an infinite antichain within either $x_1, x_2, \ldots$, or $y_1, y_2, \ldots$ which violates either X or Y being a wqo. Hence the color must be UP-UP. This shows that there is an infinite ascending sequence. ▮

4 If $(X, \preceq)$ is a wqo then $2^{\text{fin}X}$ is a wqo

Theorem 4.1 *Let $(X, \preceq)$ be a well quasi order. Let $2^{\text{fin}X}$ be the set of FINITE subsets of X. We DEFINE an order $\preceq'$ on $2^{\text{fin}X}$:*

$A \preceq' B$ if there is an injection f from A to B such that $x \preceq f(x)$.

($\emptyset \preceq' B$ is always true: use the empty function and the condition holds vacuously.) Then $(2^{\text{fin}X}, \preceq')$ is a well quasi order.

Proof:

Throughout 'smallest' means smallest CARDINALITY of a set.

Assume, BWOC, that $(2^{\text{fin}X}, \preceq')$ is a NOT a wqo.

Let A_1 be the smallest set that begins a bad sequence.

Let A_2 be the smallest set that is the second element of a bad sequence that begins with A_1

For all $i \geq 3$

Let A_i be the smallest set that is the ith element of a bad sequence that begins with $A_1, A_2, \ldots, A_{i-1}$.

Note that

$$A_1, A_2, A_3, \ldots$$

is a *minimal bad sequence.*

None of the A_i's can be empty since its a bad sequence.

Let B_i be A_i minus an element.

The elements are picked arb, however lets call the set of such elements $MINUS$.

Let $\mathcal{B} = \{B_1, B_2, \ldots\}$.

Claim: $\mathcal{B}$ with the order $\preceq'$ is a wqo

Proof of Claim: Assume, BWOC, that there is a bad sequence:

$$B_{i_1}, B_{i_2}, \ldots$$

We can assume that i_1 is the smallest index that appears (take the smallest one that appears and start there). Aside from that we DO NOT know anything about the order of the i_j's.

Look at the sequence

$$A_1, A_2, \ldots, A_{i_1-1}, B_{i_1}, B_{i_2}, \ldots$$

(NOTE we DO NOT KNOW, NOR DO WE THINK that $i_1 < i_2 < \cdots$)

We show this is a BAD sequence.

1. Since $A_1, A_2, \ldots$ is a bad sequence there will be no uptick in the first $i_1 - 1$ elements of the sequence.

2. Since $B_{i_1}, B_{i_2}, \ldots$ is a bad sequence there will be on uptick in the elements after A_{i_1-1}.

3. Assume, BWOC, that we have $i < i_j$ and $A_i \preceq' B_{i_j}$. Take the injection from A_i to B_{i_j} and view it as an injection from A_i to A_{i_j}. Hence $i < i_j$ and $A_i \preceq A_{i_j}$. Hence we have an uptick in the BAD SEQUENCE $A_1, A_2, \ldots$. This is a contradiction.

SO

$$A_1, A_2, \ldots, A_{i_1-1}, B_{i_1}, B_{i_2}, \ldots$$

is a bad sequence. Look at its i_1 element. Recall how A_{i_1} was defined:

Let A_{i_1} be the smallest set that is the i_1th element of a bad sequence that begins with $A_1, A_2, \ldots, A_{i_1-1}$.

BUT we are now looking at a bad sequence that begins with

$$A_1, A_2, \ldots, A_{i_1-1}$$

with i_1th element B_{i_1}, and $|B_{i_1}|$ is A_{i_1} with one element missing so it is SMALLER. This is a contradiction.

So $\mathcal{B}$ with $\preceq'$ is a wqo.

End of Proof of Claim

SO $\mathcal{B}$ under $\preceq'$ is a wqo

$MINUS$ under $\preceq$ is a subset of a wqo so its a wqo.

So $\mathcal{B} \times MINUS$ is a wqo.

SEQONE: $A_1, A_2, \ldots,$

View this as

SEQTWO: $(B_1, b_1), (B_2, b_2), \ldots$

Where $A_i = B_i \cap \{b_i\}$.

Since SEQTWO has an uptick, SEQONE has an uptick.

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5 The Kruskal Tree Theorem

Now that we are familiar with wqo's and minimal bad sequence arguments we can sketch the proof of the Kruskal Tree Theorem.

Theorem 5.1 *Let $(X, \leq)$ be a wqo. Let $TREEW$ be the set of trees where the nodes are labeled with elements of X. Xe define $T \preceq T'$ if you can remove vertices, remove edges, contract edges, until you get a tree T'' such that the vertices of T are $\leq$ their analogs in T'. $TREEW$ under $\preceq$ is a wqo*

Proof:

Assume, BWOC that the set of trees under minor is NOT a wqo.
Let $T_1, T_2, \ldots$ be a MINIMAL BAD SEQUENCE defined in the usual way.
None of the trees is the empty tree, so they all have a root.
Assume the root of T_i has degree d_i. For each T_i remove the root to obtain d_i trees $T_{i,1}, \ldots, T_{i,d_i}$
Let X be the set of all the $T_{i,j}$.
By the usual argument $(X, \preceq)$ is wqo.
View T_i as $(\{T_{i,1}, \ldots, T_{i,d_i}\}, \text{root of } T_i\} \in X \times X$.
Hence $T_1, T_2, \ldots$ is a sequence of elements of $X \times X$ which is a wqo, so there is an uptick.

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References

[1] J. Kruskal. Well-quasi-ordering, the tree theorem and Vazzsonyi's Conjecture. *Transactions of the American Math Society*, 95:210–225, 1960.

[2] C. Nash-Williams. On well-quasi-ordering finite trees. *Proceedings of the Cambridge Philosophical Society*, 59:833–853, 1963.