Even Perfect Numbers and Sums of Odd Cubes
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1 Introduction

Recall that a prefect number is equal to the sum of its divisors if you include 1 as a divisors. The first four prefect numbers are

- $6$
- $28 = 1^3 + 3^3$
- $496 = 1^3 + 3^3 + 5^3 + 7^3$
- $8128 = 1^3 + 3^3 + \cdots + 15^3$

Is there something interesting going on here?

We show that if $n$ is an even perfect number than there exists $k$ such that $n$ is the sum of the first $k$ odd cubes. We then discuss if this is interesting or not.

Nothing in this manuscript is due to me.

2 Needed Theorems

We rely on two well known theorems. We include their proofs for completeness For the first one we need a lemma

**Def 2.1** $\sigma(n)$ is the sum of the divisors of $n$ including both 1 and $n$. Note that a numbers is perfect iff $\sigma(n) = 2n$.

**Lemma 2.2**

1. $\sigma(ab) = \sigma(a)\sigma(b)$.

2. For all $x$, $\sigma(2^x) = 2^{x+1} - 1$. 
Proof:

1) Use the following: If \( a = \prod_{i=1}^{n} p_i^{a_i} \) and \( b = \prod_{i=1}^{n} p_i^{b_i} \) then (some of the \( a_i \) and \( b_i \)'s might be zero.)

\[
\sigma(a) = \sum_{j_1=0}^{a_{j_1}} \sum_{j_2=0}^{a_{j_2}} \cdots \sum_{j_n=0}^{a_{j_n}} \prod_{j=1}^{n} p_j^{a_{j_j}}
\]

\[
\sigma(b) = \sum_{j_1=0}^{b_{j_1}} \sum_{j_2=0}^{b_{j_2}} \cdots \sum_{j_n=0}^{b_{j_n}} \prod_{j=1}^{n} p_j^{a_{j_j}}
\]

2) Follows from part 1.

The following is the Euclid-Euler theorem since Euclid proved one direction, and Euler the other.

**Theorem 2.3** \( n \) is an even perfect number iff there exists \( p \) such that \( 2^p - 1 \) is prime and \( n = 2^{p-1}(2^p - 1) \). (The number \( p \) will also be prime since if \( 2^p - 1 \) is prime then \( p \) is a prime. We do not need this fact.)

**Proof:**

1) If \( n \) is an even perfect number then there exists \( p \) such that \( 2^p - 1 \) is prime and \( n = 2^{p-1}(2^p - 1) \).

Assume \( n \) is an even perfect number. \( n \) is even so there exists \( p \geq 2 \) and \( b \) odd such that \( n = 2^{p-1}b \). \( n \) is perfect so \( \sigma(n) = 2n = 2^p b \). By Lemma 2.2 \( \sigma(n) = \sigma(2^{p-1})\sigma(b) = (2^p - 1)\sigma(b) \).

Equating these two different expressions for \( \sigma(n) \) we obtain

\[
(\ast) \quad 2^p b = (2^p - 1)\sigma(b)
\]

Since \( 2^p - 1 \) divides \( 2^p b \) and has no factors in common with \( 2^p \), \( 2^p - 1 \) divides \( b \). Let \( b = (2^p - 1)c \). Substituting this expression for \( b \) into equation \( \ast \) yields
\[ 2^p(2^p - 1)c = (2^p - 1)\sigma(b) \]

\[ 2^p c = \sigma(b) \]

Since \( c \) divides \( b \) and (of course) \( b \) divides \( b \), \( \sigma(b) \geq b + c \). Hence

\[ 2^p c = \sigma(b) \geq b + c = (2^p - 1)c + c = 2^p c \]

Hence \( \sigma(b) = b + c \). Since \( \sigma(b) \geq b + 1 \) (\( b \) and 1 both divide \( b \)) we have \( c = 1 \) so \( \sigma(b) = b + 1 \), hence \( b = 2^p - 1 \), \( b \) is prime, and \( n = 2^p(2^p - 1) \).

2) If \( n = 2^{p-1}(2^p - 1) \) where \( 2^p - 1 \) is prime then \( n \) is perfect.

\[ \sigma(n) = \sigma(2^{p-1}(2^p - 1)) = \sigma(2^{p-1})\sigma(2^p - 1) = (2^p - 1)(1 + (2^p - 1)) = 2^p(2^p - 1) = 2 \times 2^{p-1}(2^p - 1) = 2n. \]

**Theorem 2.4** For all \( m \geq 1 \), \( \sum_{i=0}^{m-1}(2i + 1)^3 = m^2(2m^2 - 1) \).

**How you would derive this**

We are not going to do a standard proof by induction. Instead we discuss how you might derive this by hand with a minimum of calculation.

Since \( \sum_{i=0}^{m-1}(2i + 1)^3 \) is approximately \( \int_{1}^{m-1}(2x + 1)^3 dx \) we can guess that the lead term is a polynomial of degree 4 with lead term \( 2m^4 \). So we need to find \( b, c, d, e \) such that

\[ \sum_{i=0}^{m-1}(2i + 1)^3 = 2m^4 + bm^3 + cm^2 + dm + e \]
Since when $m = 0$ the sum is 0 we get $e = 0$.

From here there are two ways to proceed: (1) plug in $m = 1, 2, 3$ to get three linear equations in three variables. (2) do a proof by induction and see what the proof forces $b, c, d$ to be.

**End of How you would derive this**

### 3 The Main Theorem

**Theorem 3.1** If $n$ is an even perfect number then there exists $m$ such that $n$ is the sum of the first $m - 1$ odd cubes.

**Proof:**

By Theorem 2.3 there exists $p$ such that $2^p - 1$ is prime and $n = 2^{p-1}(2^p - 1)$. Let $m - 1 = 2^{(p-1)/2}$.

By Theorem 2.4

\[
\sum_{i=0}^{m-1} (2i + 1)^3 = m^2(2m^2 - 1) = (2^{(p-1)/2})^2(2 \times (2^{(p-1)/2})^2 - 1) = 2^{p-1} \times (2^p - 1) = n
\]

We never used that $2^p - 1$ is prime. We never used that $n$ is perfect. We did use that $p$ is odd (so that $p - 1$ is even). Hence we have the following theorem.

**Theorem 3.2** If $n$ is of the form $2^{p-1}(2^p - 1)$ where $p$ is odd then $n$ is the sum of the first $(p-1)/2$ odd squares.

Even though this is more general it somehow sounds less interesting.