Even Perfect Numbers and Sums of Odd Cubes

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1 Introduction

Recall that a prefect number is equal to the sum of its divisors if you include 1 as a divisors. The first four prefect numbers are

6

$28 = 1^3 + 3^3$

$496 = 1^3 + 3^3 + 5^3 + 7^3$

$8128 = 1^3 + 3^3 + \cdots + 15^3$

Is there something interesting going on here?

We show that if n is an even perfect number than there exists k such that n is the sum of the first k odd cubes. We then discuss if this is interesting or not.

Nothing in this manuscript is due to me.

2 Needed Theorems

We rely on two well known theorems. We include their proofs for completeness For the first one we need a lemma

Def 2.1 $\sigma(n)$ is the sum of the divisors of n including both 1 and n. Note that a numbers is perfect iff $\sigma(n) = 2n$.

Lemma 2.2

1. $\sigma(ab) = \sigma(a)\sigma(b)$.

2. *For all* x, $\sigma(2^x) = 2^{x+1} - 1$.

Proof:

1) Use the following: If $a = \prod_{i=1}^{n} p_i^{a_i}$ and $b = \prod_{i=1}^{n} p_i^{b_i}$ then (some of the a_i and b_i's might be zero.)

$$\sigma(a) = \sum_{j_1=0}^{a_{i_1}} \sum_{j_2=0}^{a_{j_2}} \cdots \sum_{j_n=0}^{a_{j_n}} \prod_{j=1}^{n} p_i^{a_{j_i}}$$

$$\sigma(b) = \sum_{j_1=0}^{b_{j_1}} \sum_{j_2=0}^{b_{i_2}} \cdots \sum_{j_n=0}^{bj_n} \prod_{j=1}^{n} p_i^{a_{j_i}}$$

2) Follows from part 1.

▮

The following is the Euclid-Euler theorem since Euclid proved one direction, and Euler the other.

Theorem 2.3 *n is an even perfect number iff there exists p such that $2^p - 1$ is prime and $n = 2^{p-1}(2^p - 1)$. (The number p will also be prime since if $2^p - 1$ is prime then p is a prime. We do not need this fact.)*

Proof:

1) If n is an even perfect number then here exists p such that $2^p - 1$ is prime and $n = 2^{p-1}(2^p - 1)$.

Assume n is an even perfect number. n is even so there exists $p \geq 2$ and b odd such that $n = 2^{p-1}b$. n is perfect so $\sigma(n) = 2n = 2^p b$. By Lemma 2.2 $\sigma(n) = \sigma(2^{p-1})\sigma(b) = (2^p - 1)\sigma(b)$. Equating these two different expressions for $\sigma(n)$ we obtain

$$(*) \qquad 2^p b = (2^p - 1)\sigma(b).$$

Since $2^p - 1$ divides $2^p b$ and has no factors in common with 2^p, $2^p - 1$ divides b. Let $b = (2^p - 1)c$. Substituting this expression for b into equation $*$ yields

$$2^p(2^p-1)c = (2^p-1)\sigma(b)$$

$$2^p c = \sigma(b)$$

Since c divides b and (of course) b divides b, $\sigma(b) \geq b + c$. Hence

$$2^p c = \sigma(b) \geq b + c = (2^p-1)c + c = 2^p c$$

Hence $\sigma(b) = b + c$. Since $\sigma(b) \geq b + 1$ (b and 1 both divide b) we have $c = 1$ so $\sigma(b) = b + 1$, hence $b = 2^p - 1$, b is prime, and $n = 2^p(2^p-1)$.

2) If $n = 2^{p-1}(2^p-1)$ where $2^p - 1$ is prime then n is perfect.

$$\sigma(n) = \sigma(2^{p-1}(2^p-1)) = \sigma(2^{p-1})\sigma(2^p-1) = (2^p-1)(1+(2^p-1)) = 2^p(2^p-1) = 2\times 2^{p-1}(2^p-1) = 2n.$$

▮

Theorem 2.4 *For all* $m \geq 1$, $\sum_{i=0}^{m-1}(2i+1)^3 = m^2(2m^2-1)$.

How you would derive this

We are not going to do a standard proof by induction. Instead we discuss how you might derive this by hand with a minimum of calculation.

Since $\sum_{i=0}^{m-1}(2i+1)^3$ is approximately $\int_1^{m-1}(2x+1)^3 dx$ we can guess that the lead term is a polynomial of degree 4 with lead term $2m^4$. So we need to find b, c, d, e such that

$$\sum_{i=0}^{m-1}(2i+1)^3 = 2m^4 + bm^3 + cm^2 + dm + e$$

Since when $m = 0$ the sum is 0 we get $e = 0$.

From here there are two ways to proceed: (1) plug in $m = 1, 2, 3$ to get three linear equations in three variables. (2) do a proof by induction and see what the proof forces b, c, d to be.

End of How you would derive this

3 The Main Theorem

Theorem 3.1 *If n is an even perfect number then there exists m such that n is the sum of the first $m - 1$ odd cubes.*

Proof:

By Theorem 2.3 there exists p such that $2^p - 1$ is prime and $n = 2^{p-1}(2^p - 1)$. Let $m - 1 = 2^{(p-1)/2}$.

By Theorem 2.4

$$\sum_{i=0}^{m-1}(2i+1)^3 = m^2(2m^2-1) = (2^{(p-1)/2})^2(2 \times (2^{(p-1)/2})^2 - 1) = 2^{p-1} \times (2^p - 1) = n$$

∎

We never used that $2^p - 1$ is prime. We never used that n is perfect. We did use that p is odd (so that $p - 1$ is even). Hence we have the following theorem.

Theorem 3.2 *If n is of the form $2^{p-1}(2^p - 1)$ where p is odd then n is the sum of the first $(p-1)/2$ odd squares.*

Even though this is more general it somehow sounds less interesting.