Approximating π, and $\ln(2)$ Using Integrals
by William Gasarch

1 What Here Is New? Not Much

As I was finishing this manuscript, I came across an excellent paper by Frits Beukers [1] which has everything I have about π and much more. He does not discuss $\ln(2)$. The only change I made was to add this paragraph and use his terminology about approximating irrationals in the sections where I approximate π and $\ln(2)$.

2 An Integral that Shows $\frac{22}{7} \sim \pi$

Problem A-1 on the Putnam Exam in 1968 was:

Prove that $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \frac{22}{7} - \pi$.

Here is the solution:

$$\frac{x^4(1-x)^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$$

Hence

$$\int \frac{x^4(1-x)^4}{1+x^2}dx = \int_0^1 x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}dx = \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x - 4\arctan(x)\right]_0^1$$

$$= \frac{22}{7} - 4\arctan(1) - 0 = \frac{22}{7} - \pi$$

Can we show how good an approximation $\frac{22}{7}$ is to π using this integral? We give a lemma that will be useful for this and later in the paper.

Lemma 2.1 *For all n,*

$$0 \le \int_0^1 \frac{x^n(1-x)^n}{1+x^2}dx \le \frac{1}{2^{2n}}.$$

Proof: Since the integral is non-negative on $[0,1]$, we have the lower bound.

Since

$$\int_0^1 \frac{x^n(1-x)^n}{1+x^2}dx \leq \int_0^1 x^n(1-x)^n dx \leq 1 \times \max_{0\leq x\leq 1} x^n(1-x)^n = \frac{1}{2^{2n}}$$

we have the upper bound. ▮

By letting $n=4$ we find that $0 \leq \frac{22}{7} - \pi \leq \frac{1}{2^8}$.

In this paper we look at how $\int_0^1 \frac{x^n(1-x)^n}{1+x^2}dx$ can help us approximate π and $\ln(2)$ with error bounds.

3 The Non-Polynomial Part of $\frac{x^n(1-x)^n}{x^2+1}$

Let $f_n(x) = \frac{x^n(1-x)^n}{x^2+1}$. To integrate this you would use partial fractions to express it as a polynomial plus a non-polynomial part. The non-polynomial part ends up being a linear combination of $\frac{1}{x^2+1}$, which has integral $\arctan(x)$, and $\frac{x}{x^2+1}$, which has integral $\frac{1}{2}\ln(x^2+1)$. It is this non-polynomial part that leads to π and $\ln(2)$.

We give a table of the non-polynomial parts of $f_n(x)$, along with the integral from 0 to 1 of that non-polynomial part.

n	nonpoly	$\int$	$\int_0^1$
0	$\frac{1}{x^2+1}$	$\arctan(x)$	$\frac{\pi}{4}$
1	$\frac{1}{x^2+1} + \frac{x}{x^2+1}$	$\arctan(x) + \frac{1}{2}\ln(x^2+1)$	$\frac{\pi}{4} + \frac{1}{2}\ln(2)$
2	$\frac{2x}{x^2+1}$	$\ln(x^2+1)$	$\ln(2)$
3	$-\frac{2}{x^2+1} + \frac{2x}{x^2+1}$	$-2\arctan(x) + \ln(x^2+1)$	$-\frac{\pi}{2} + \ln(2)$
4	$-\frac{4}{x^2+1}$	$-4\arctan(x)$	$-\pi$
5	$-\frac{4}{x^2+1} - \frac{4x}{x^2+1}$	$-4\arctan(x) - 2\ln(x^2+1)$	$-\pi - 2\ln(2)$
6	$-\frac{8x}{x^2+1}$	$-4\ln(x^2+1)$	$-4\ln(2)$
7	$\frac{8}{x^2+1} - \frac{8x}{x^2+1}$	$8\arctan(x) - 4\ln(x^2+1)$	$2\pi - 4\ln(2)$
8	$\frac{16}{x^2+1}$	$16\arctan(x)$	4π
9	$\frac{16}{x^2+1} + \frac{16x}{x^2+1}$	$16\arctan(x) + 8\ln(x^2+1)$	$4\pi + 8\ln(2)$
10	$\frac{32x}{x^2+1}$	$16\ln(x^2+1)$	$16\ln(2)$
11	$-\frac{32}{x^2+1} + \frac{32x}{x^2+1}$	$-32\arctan(x) + 16\ln(x^2+1)$	$-8\pi + 16\ln(2)$
12	$-\frac{64}{x^2+1}$	$-64\arctan(x)$	-8π
13	$-\frac{64}{x^2+1} - \frac{64x}{x^2+1}$	$-64\arctan(x) - 32\ln(x^2+1)$	$-16\pi - 32\ln(2)$
14	$-\frac{128x}{x^2+1}$	$-64\ln(x^2+1)$	$-64\ln(2)$
15	$-\frac{128x}{x^2+1} + \frac{128}{x^2+1}$	$-64\arctan(x) + 128\arctan(x)$	$-64\ln(2) + 32\pi$

4 The Non-Polynomial Part is Always ...

We show that the behavior in the table of the last section always holds.

Theorem 4.1 *There exists a sequence of polynomials $p_0, p_1, p_2, \ldots$ such that*

1. $\int \frac{x^{8k}(1-x)^{8k}}{x^2+1} dx = p_{8k}(x) + 2^{4k}\arctan(x)$.

2. $\int \frac{x^{8k+1}(1-x)^{8k+1}}{x^2+1} dx = p_{8k+1}(x) + 2^{4k}\arctan(x) + 2^{4k-1}\ln(x^2+1)$.

3. $\int \frac{x^{8k+2}(1-x)^{8k+2}}{x^2+1} dx = p_{8k+2}(x) + 2^{4k}\ln(x^2+1)$.

4. $\int \frac{x^{8k+3}(1-x)^{8k+3}}{x^2+1}dx = p_{8k+3}(x) - 2^{4k+1}\arctan(x) + 2^{4k}\ln(x^2+1)$.

5. $\int \frac{x^{8k+4}(1-x)^{8k+4}}{x^2+1}dx = p_{8k+4}(x) - 2^{4k+2}\arctan(x)$.

6. $\int \frac{x^{8k+5}(1-x)^{8k+5}}{x^2+1}dx = p_{8k+5}(x) - 2^{4k+2}\arctan(x) - 2^{4k+1}\ln(x^2+1)$.

7. $\int \frac{x^{8k+6}(1-x)^{8k+6}}{x^2+1}dx = p_{8k+6}(x) - 2^{4k+2}\ln(x^2+1)$.

8. $\int \frac{x^{8k+7}(1-x)^{8k+7}}{x^2+1}dx = p_{8k+7}(x) - 2^{4k+3}\arctan(x) + 2^{4k+2}\ln(x^2+1)$.

Proof: Let $0 \le a \le 7$ and $k \in \mathbb{N}$. We seek the non-polynomial part of $\frac{x^{8k+a}(1-x)^{8k+a}}{x^2+1}$. Hence we want to know A, B such that there is a polynomial p with

$$\frac{x^{8k+a}(1-x)^{8k+a}}{x^2+1} = p(x) + \frac{Ax+B}{1+x^2}$$

$$\Longrightarrow \frac{x^{8k+a}(1-x)^{8k+a} - Ax - B}{x^2+1} = p(x)$$

In order for this to be polynomial, x^2+1 must divide the numerator. So i and $-i$ must be roots of the numerator. Since i and $-i$ are conjugate, we only need that i is a root. Thus we need A, B such that the following quantity is 0.

$$i^{8k+a}(1-i)^{8k+a} - Ai - B = i^a 2^{4k}(1-i)^a - Ai - B$$

There are eight cases: $0 \le a \le 7$. We do the $a = 0$ and $a = 1$ cases. The rest are similar.

Case $a = 0$: We need $2^{4k} - Ai - B = 0$, so $A = 0$ and $B = 2^{4k}$. Hence there is some polynomial p such that

$$\frac{x^{8k}(1-x)^{8k}}{x^2+1} = p(x) + \frac{2^{4k}}{1+x^2}.$$

Let $p_{8k}(x)$ be the integral of $p(x)$. We have

$$\int \frac{x^{8k}(1-x)^{8k}}{x^2+1}dx = p_{8k}(x) + 2^{4k}\arctan(x).$$

Case $a = 1$: We need $i2^{4k}(1-i) - Ai - B = 0$, so $A = 2^{4k}$ and $B = 2^{4k}$. Hence there is some polynomial p such that

$$\frac{x^{8k+1}(1-x)^{8k+1}}{x^2+1} = p(x) + \frac{2^{4k}x}{1+x^2} + \frac{2^{4k}}{1+x^2}.$$

Let $p_{8k+1}(x)$ be the integral of $p(x)$. We have

$$\int \frac{x^{8k+1}(1-x)^{8k+1}}{x^2+1}dx = p_{8k+1}(x) + 2^{4k-1}\ln(x^2+1) + 2^{4k}\arctan(x).$$

$\blacksquare$

5 Approximating π and $\ln(2)$

We use Theorem 4.1 to approximate π. By similar techniques one can approximate $\ln(2)$.

1. Input n (we want π within $\frac{1}{2^n}$) and indicate if you want your approximation to be *bigger* or *smaller* than π.

2. Let $k = \lceil \frac{n}{20} \rceil$. If you requested *bigger*, then use $8k$ in the next step (as we do). If you requested *smaller*, then use $8k+4$ in the next step (which we leave as an exercise).

3. Find $\int \frac{x^{8k}(1-x)^{8k}}{x^2+1}dx$. It will be of the form $p(x) + 2^{4k}\arctan(x)$. By Lemma 2.1

$$0 \le \int_0^1 \frac{x^{8k}(1-x)^{8k}}{x^2+1}dx = p(1) + 2^{4k}\pi - p(0) \le \frac{1}{2^{16k}}$$

,

$$0 \le \pi - \frac{p(0)-p(1)}{2^{4k}} \le \frac{1}{2^{20k}} \le \frac{1}{2^n}$$

4. Output $\frac{p(0)-p(1)}{2^{4k}}$.

6 Approximating π: Actual Numbers

We used our methods on $n \equiv 0 \pmod 4$ to approximate π. Frits Beukers [1] measures how good a rational approximation $\frac{p}{q}$ to irrational α is by the largest M such that

$$\left|\frac{p}{q} - \alpha\right| \le \frac{1}{q^M}.$$

All irrationals have a rational approximation with $M = 2$; however, it is often hard to find it. Let's see how we do with our approximation to π. Spoiler Alert: Not that well.

a	$0 \le \int_0^1 \frac{x^n(1-x)^n}{1+x^2} dx \le \frac{1}{2^{2a}}$	so	M
0	$0 \le 2^{-2}\pi \le \frac{1}{2^0}$	$0 \le \pi \le 2^2$	
4	$0 \le \frac{22}{7} - 2^0\pi \le \frac{1}{2^8}$	$0 \le \frac{22}{7} - \pi \le \frac{1}{2^8}$	
8	$0 \le 2^2\pi - \frac{188684}{15015} \le \frac{1}{2^{16}}$	$0 \le \pi - \frac{188684}{60060} \le \frac{1}{2^{20}}$	

We consider an example with a large $n \equiv 0 \pmod 4$. We look at $n = 48$.

$$0 \le \int_0^1 \frac{x^{48}(1-x)^{48}}{1+x^2} dx \le \frac{1}{2^{96}}$$

Using Wolfram Alpha with $n = 48$ we quickly obtained:

$$0 \le 2^{22}\pi - \frac{21354261181403767106687606946881637275344}{1620596038387196643453445745547157 5} \le \frac{1}{2^{96}}.$$

Hence we have

$$\left| 0 \le \pi - \frac{21354261181403767106687606946881637275344}{6797272446191572430423361304331424890880 0} \right| \le \frac{1}{2^{118}}.$$

This yields $M \sim 0.87$.

7 Approximating $\ln(2)$: Actual Numbers

We used our methods on $n \equiv 0 \pmod 4$ to approximate π. We use M to measure the closeness of the approximation as in the last section.

a	$0 \le \int_0^1 \frac{x^n(1-x)^n}{1+x^2} dx \le \frac{1}{2^{2a}}$	so	M
2	$0 \le \ln(2) - \frac{2}{3} \le 2^4$	$0 \le \ln(2) - \frac{2}{3} \le \frac{1}{2^4}$	
6	$0 \le \frac{38429}{13860} - 4\ln(2) \le \frac{1}{2^{12}}$	$0 \le \frac{38429}{55440} - \ln(2) \le \frac{1}{2^{14}}$	

8 $\int_0^1 \frac{x^n(1-x)^n}{1+x^2} dx$

n	answer	so	approx
0	$\frac{\pi}{4} - 0$	$\pi > 0$	0
1	$-1 + \frac{\pi}{4} + \frac{1}{2}\ln(2)$	$\pi + 2\ln(2) > 4$	4
2	$\ln(2) - \frac{2}{3}$	$\ln(2) > \frac{2}{3}$	0.66666666666
3	$\frac{53}{60} - \frac{\pi}{2} + \ln(2)$	$\pi - 2\ln(2) < \frac{53}{30}$	1.76666666666
4	$\frac{22}{7} - \pi$	$\pi < \frac{22}{7}$	3.14285714286
5	$\frac{11411}{2520} - 2\ln(2) - \pi$	$\pi + 2\ln(2) < \frac{11411}{2520}$	4.52817460317
6	$\frac{38429}{13860} - 4\ln(2)$	$\ln(2) < \frac{38429}{55440}$	0.69316378066
7	$-\frac{421691}{120120} - 4\ln(2) + 2\pi$	$\pi - 2\ln(2) > \frac{421691}{240240}$	1.75529054279
8	$4\pi - \frac{188684}{15015}$	$\pi > \frac{188684}{60060}$	3.14159174159
9	$-\frac{17069771}{942480} + 4\pi + 8\ln(2)$	$\pi + 2\ln(2) > \frac{17069771}{3769920}$	4.52788679866
10	$16\ln(2) - \frac{1290876029}{116396280}$	$\ln(2) > \frac{1290876029}{1862340480}$	0.69314716769
11	$\frac{817240769}{58198140} - 8\pi + 16\ln(2)$	$\pi - 2\ln(2) < \frac{817240769}{465585120}$	1.75529829862
12	$\frac{431302721}{8580495} - 16\pi$	$\pi < \frac{431302721}{137287920}$	3.14159265433
13	$\frac{1939467473639}{26771144400} - 16\pi - 32\ln(2)$	$\pi + 2\ln(2) < \frac{1939467473639}{428338310400}$	4.52788701489
14	$\frac{356281790621}{8031343320} - 64\ln(2)$	$\ln(2) < \frac{356281790621}{514005972480}$	0.69314718057
15	$-\frac{130823901842567}{2329089562800} + 32\pi - 64\ln(2)$	$\pi - 2\ln(2) > \frac{130823901842567}{2329089562800\times 32}$	1.75529829246
16	$64\pi - \frac{5930158704872}{29494189725}$	$\pi > \frac{5930158704872}{29494189725\times 64}$	3.14159265359
17	$-\frac{1549850007762613}{5348279736800} + 64\pi + 128\ln(2)$	$\pi + 2\ln(2) > \frac{1549850007762613}{5348279736800\times 128}$	2.26394350735
18	$256\ln(2) - \frac{12811893190532663}{72201776446800}$	$\ln(2) > \frac{12811893190532663}{72201776446800\times 256}$	0.69314718056
19	$\frac{600220061655474431}{2671465728531600} - 128\pi + 256\ln(2)$	$\pi - 2\ln(2) < \frac{600220061655474431}{2671465728531600\times 128}$	1.75529829247
20	$\frac{26856502742629699}{33393321606645} - 256\pi$	$\pi < \frac{26856502742629699}{33393321606645\times 256}$	3.14159265359
21	$\frac{8464040862393468127}{7302006324653040} - 256\pi - 512\ln(2)$	$\pi + 2\ln(2) < \frac{8464040862393468127}{7302006324653040\times 256}$	4.52788701471
22	$\frac{334293041884658943643}{470979407940121080} - 1024\ln(2)$	$\ln(2) < \frac{334293041884658943643}{470979407940121080\times 1024}$	0.69314718056
23	$-\frac{1410917291598131585159}{1569931359800403600} + 512\pi - 1024\ln(2)$	$\pi - 2\ln(2) > \frac{1410917291598131585159}{1569931359800403600\times 512}$	1.75529829247

References

[1] F. Beukers. A rational approach to π. *Nieuw Archief voor Wiskunde*, 5:372–379, 2000.