Breakthrough in My Favorite Open Problem of Mathematics: Chromatic Number of the Plane

Alexander Soifer

> *[I] can't offer money for nice problems of other people because then I will really go broke… It is a very nice problem. If it were mine, I would offer $250 for it.*
>
> – Paul Erdős
>
> Boca Raton, February 1992

1. Chromatic Number of the Plane: The Problem

On behalf of all enthusiasts of this problem, let me express gratitude to my friend, the late Edward Nelson, who created this problem at a tender age of 18 in November 1950:

> *What is the smallest number of colors sufficient for coloring the plane in such a way that no two points of the same color are unit distance apart?*

This number is called *the chromatic number of the plane* and is denoted by χ. In 1961, the Swiss geometer Hugo Hadwiger admitted that he was not the author of the problem, even though the name "Harwiger-Nelson" got stuck to the problem, just as Cardano did not author the Cardano Formula, and Pythagoras Theorem was known a millennium before the great Greek was born. Such is life with credits in mathematics. Right at problem's birth, Eddie Nelson determined the lower bound of 4, and his 20-year old friend John Isbell, 20 figured out the upper bound of 7:

$$\chi = 4, \text{ or } 5, \text{ or } 6, \text{ or } 7$$

A very broad spread. Which one is the value of χ? Paul Erdős thought $\chi \geq 5$. On May 28, 2009, during the DIMACS Ramsey Theory International Workshop that I organized on request of DIMACS Director Fred Roberts, I asked the distinguished audience to determine the chromatic number of the plane by democratic means of a vote. Except for one young lady voting for 6, and I voting for 7, the rest of the workshop participants *equally* split between 4 (including Peter D. Johnson Jr.), and 5 (including Ron Graham). I was therefore able to determine *the democratic value of the chromatic number of the plane*: 4.5.

Through the years, Ronald L. Graham offered to reward the progress on this problem (I quote here from [Soi]):

On Saturday, May 4, 2002, which by the way was precisely my friend Edward Nelson's 70th birthday, Ronald L. Graham gave a talk on Ramsey Theory at the Massachusetts Institute of Technology for about 200 participants of the USA Mathematical Olympiad. During the talk he offered $1,000 for the first proof or disproof of what he called, after Nelson, "Another 4-Color Conjecture." The talk commenced at 10:30 AM (I attended the talk and took notes).

Another 4-Color $1000 Conjecture 3 (Graham, May 4, 2002). Is it possible to 4-color the plane to forbid a monochromatic distance 1?

In August 2003, in his talk *What is Ramsey Theory?* at the Mathematical Sciences Research Institute in Berkeley, California [Gra1], Graham asked for more work for $1000:

$1000 Open Problem 4 (Graham, August 2003). Determine the value of the chromatic number χ of the plane.

It seems that Ron came to believe that the chromatic number of the plane takes on an intermediate value, between its known boundaries, for in his two latest surveys [Gra2], [Gra3], he offers the following open problems:

$100 Open Problem 5 (Graham [Gra2], [Gra3]). Show that $\chi \geq 5$.

$250 Open Problem 6 (Graham [Gra2], [Gra3]). Show that $\chi \leq 6$.

2. Aubrey D.N.J. de Grey's Breakthrough

In 68 years of the problem's life, many fine mathematicians obtained many beautiful results in special circumstances [Soi]. However, no progress has occurred in the general case until Aubrey de Grey succeeded in a joint effort of his imagination and a clever computer program he wrote for this purpose. On January 16, 2018, de Grey sent me the first version, followed by a corrected one on April 7, 2018 with the following introductory note:

> I append a copy of an email I sent you in January, which you may have overlooked. It's good that you did, because the graph that I told you about is in fact 4-colorable after all, and my failure to discover this was due to a bug in my code. However, I'm pleased to report that after fixing the bug I was rapidly able, using the same basic approach, to find somewhat larger unit-distance graphs that do indeed have no 4-colourings. My confidence that my code is not still lying to me arises largely from the fact that the 4-colouring of the earlier graph was found

> by Dr. Robert Hochberg, to whom I wrote at the same time as you; he became interested enough to spend time writing code that could test quite large graphs, and he has not found a 4-colouring of the (progressively smaller, but still four-digit) graphs I have been discovering since January even though his code can 4-colour all his previous attempts at 5-chromaticity in seconds. We appreciate that this is not a proof… but it makes us feel good enough about the graphs that I have now written up the discovery as a paper. I have just submitted it to the arXiv and it is scheduled to go live on Monday. I still very much hope that you will be inclined to consider it for publication in *Geombinatorics*; I am in absolute awe of your 2008 book and I hope that this might serve as some sort of mark of my gratitude.

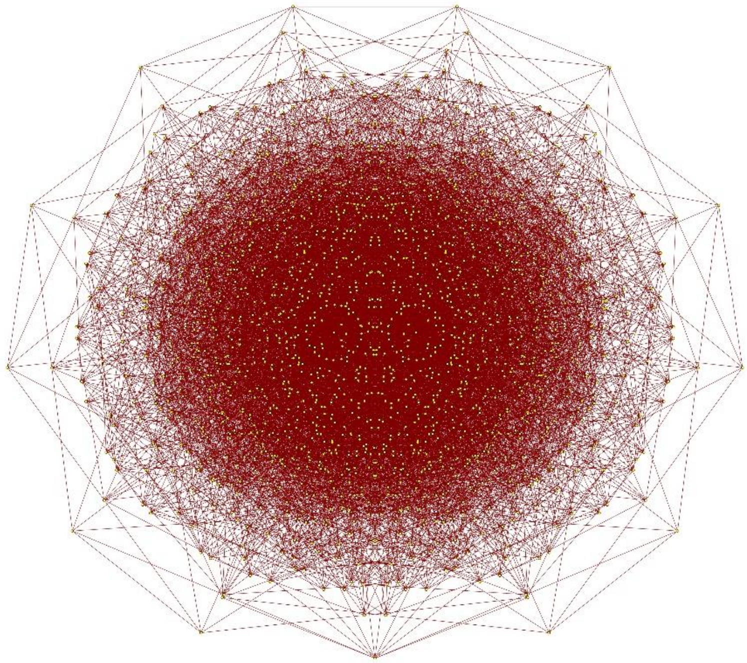
Previous pursuits of a 5-chromatic unit-distance graph (see O'Donnell [O'D], for example) were based on an idea that unit-distance graphs without 3-cycles could be easily bent and embedded in the plane. No triangles, of course, means no Mosers' Spindles. De Grey goes the opposite way: he floods his construction with lots of Mosers' Spindles. His goal is to force a certain coloring of small number of vertices and then create contradictions to those forced colorations.

He constructs a unit-distance graph on 20,425 vertices. Can one check whether it is 4-colorable? Normally one would not even try. But de Grey bravely goes for it, and with a clever use of specific properties of his graph succeeds in verifying that this giant graph is indeed not 4-colorable. Hence, we arrive at

De Grey's Theorem. $\chi \geq 5$.

De Grey then dramatically reduces the size of his 5-chromatic unit-distance graph – see it in his essay in this issue. It also decorates the cover of this journal.

De Grey's Example. There is a 5-chromatic unit-distance graph on 1581 vertices.



De Grey’s 5-chromatic unit-distance graph on 1581 vertices.

With this size, the verification becomes within the reach of his computer program. What does Aubrey do next? I knew many colleagues, who would keep their approach in secret, or worse, would publish hardly comprehensible description – in order to position themselves ahead of the competition. Aubrey de Grey is a true scholar. He does not wish to compete with others, but rather invites them to join in to conquer mathematics herself. He succeeds in commencing a Polymath project where new blood is attracted to try their twist on the problem. And try they do.

In 1991, I started *Geombinatorics*, jointly with the great geometer Branko Grünbaum, in the style of Paul Erdős’ problem-posing talks and essays. I found existing mathematical journals to be like cemeteries for honorable burials of finished research. My goal was to publish research in progress, so that people can join in the efforts. Clearly, *Geombinatorics* has been a precursor to

Polymath-type blogs. With your contributions and enthusiasm, it will continue to be our meeting place, the melting pot of ideas.

3. Marijn J.H. Heule

Marijn Heule, a virtuoso computer scientist, who does not always rely on existing software but rather creates his own. We are fortunate that he became excited about CNP problem, for he has produced a series of world records for the smallest known 5-chromatic unit-distance graphs, with apparently no one able to compete with him in this endeavor. Let us document for posterity the succession of his 5-chromatic unit-distance graph records:

Heule's Example 1. 874 vertices and 4461 edges. Created on April 14, 2018

Heule's Example 2. 826 vertices and 4273 edges. Created on April 16, 2018

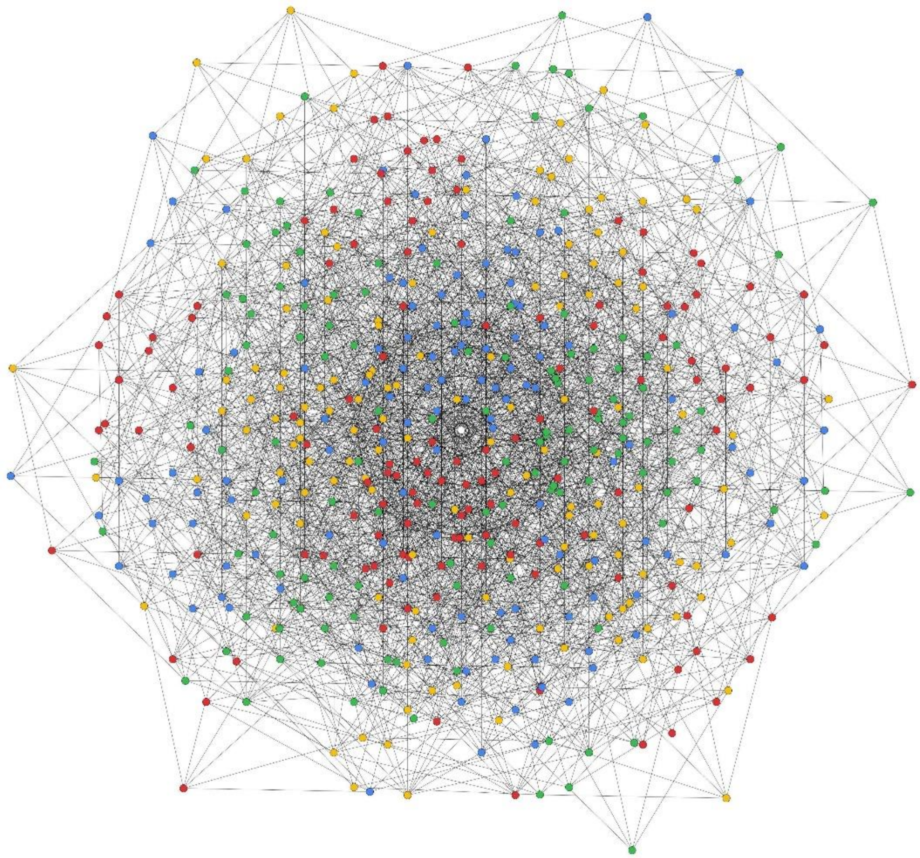
Heule's Example 3. 803 vertices and 4144 edges. Created on April 30, 2018

Heule's Example 4. 633 vertices and 3166 edges. Created on May 6, 2018

Heule's Example 5. 610 vertices and 3000 edges. Created on May 14, 2018

Marijn Heule ends up – so far – with his new world record:

Heule's Example 6: 533 vertices with 2722 edges. Created on May 18, 2018



Heule's 5-chromatic unit-distance graph on 533 vertices with 2722 edges

Paul Erdős' reply to me regarding Appel-Haken proof of the Four-Color Conjecture, is applicable here too:

> *I prefer a computer-free proof of the Four-Color Conjecture, but I am willing to accept the Appel-Haken solution. Beggars cannot be choosers.*

The computer-free proof is preferred only because we are a curious bunch and a human-checkable proof *may* shed light on why things are the way they are.

4. Geoffrey Exoo and Dan Ismailescu

And then we have a team of a *Geombinatorics Editor* Geoffrey Exoo and Dan Ismailescu. Readers of *Geombinatorics* recall that this duo not only created a 4-chromatic unit-distance graph of girth 4 on 17 vertices, but also proved that 17 is best possible, thus settling [EI] an important open problem 15.4 I posed in [Soi].

Geoffrey tells me – and I certainly trust him – that they had their 5-chromatic unit-distance graph at about the same time as Aubrey de Grey, in January 2018. They then took their time to improve construction, reduce its size, and … missed the boat. Having acknowledged – as they ought – de Grey's priority, they published their graph elsewhere. In this issue, they are building tools clearly aimed at constructing a 6-chromatic unit-distance graph. I can't wait to witness its birth.

5. The Road Ahead

We are in search of Beauty, existing in Nature or invented by us. This Special Issue of *Geombinatorics*, opening its 28th year, I publish in full color to demonstrate that the graphs created by our authors not only carry a great deal of mathematical significance, but are also strikingly beautiful.

I am sure the size of the smallest 5-chromatic unit-distance graph will go down, perhaps to the neighborhood of ca. 200 vertices. There is a new energy in the air, promising a 6-chromatic unit-distance graph in not too distant future. Some scholars tell me that they commence working on reducing the upper bound of χ from 7 to 6, and the upper bound for the chromatic number of 3-space from 15 to 14. I stand by my old conjectures:

CNP Conjecture for the Plane [Soi]. $\chi = 7$.

CNP Conjecture for E^3 [Soi]. $\chi\left(E^3\right) = 15$.

Genius things are often simple. In the end, simplicity is what we are after. And so, here is my old simple conjecture for the general case:

CNP Conjecture for the Euclidean *n*-space E^n [Soi]**:**

$$\chi\left(E^n\right) = 2^{n+1} - 1.$$

When will we learn the exact value of CNP? Niels Bohr chuckled "It is hard to predict, especially the future." And yet in 1991, the late Victor Klee & Stan Wagon [KW] braved the prediction:

> *If Problem 8 [CNP] takes that long to settle [as the Four-Color Conjecture], we should know the answer by the year 2084.*

There is still time remaining before the year 2084, the century pass George

Orwell's favorite year of 1984. And so, Ladies and Gentlemen, start your engines!

This was a concise version of my essay, which will appear in the Special Issue XXVIII(1) of *Geombinatorics*, July 2018, 5–17, dedicated to 5-chromatic unit distance graphs. It includes contributions by Aubrey D.N.J. de Grey, Marijn J.H. Heule, and Geoffrey Exoo and Dan Ismailescu.

Bibliography

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