Review [1] of
The Mathematical Coloring Book:
Mathematics of Coloring and the Colorful Life of its Creators
by Alexander Soifer
Springer, 2009 $60.00
approx 600 pages, Hardcover

Review by
William Gasarch `gasarch@cs.umd.edu`

1 Introduction

I first had the pleasure of meeting Alexander Soifer at one of the *Southeastern International Conferences on Combinatorics, Computing, and Graph Theory.* If I was as careful a historian as he is, I would know which one. Over lunch he told me about van der Waerden's behavior when he was living as a Dutch Citizen in Nazi Germany. Van der Waerden later claimed that he opposed the firing of Jewish professors. Soifer explained to me that in 1933 the German government passed a law requiring Universities to fire all Jewish professors *unless they were veterans of WW I* (there were other exceptions also). Van der Waerden protested that veterans were being fired, in violation of the law. So he was objecting to *the law not being carried out properly* and not to *the law itself.* Alex told me that the full story would soon appear in a book he was writing on Coloring Theorems. I couldn't tell if the book would be a math book or a history book. It is both.

The second time I met Alex was at the next *Southeastern International Conference on Combinatorics, Computing, and Graph Theory.* Alex gave a talk on the following:

1. Prove that for any 2-coloring of the plane there are two points an inch apart that are the same color. (This is easy: I was able to do it in 2 minutes.)

2. Prove that for any 3-coloring of the the plane there are two points an inch apart that are the same color. (This is easy: I was able to do it in 3 minutes.)

3. Prove that there is a 7-coloring of the plane such that for all points p, q that are an inch apart, p and q are different colors. (This is easy: I was able to do it in 7 minutes.)

4. Find the number χ such that (1) for any $(\chi - 1)$-coloring of the plane there will be two points an inch apart that are the same color, and (2) there exists a χ-coloring of the plane such that for all points p, q that are an inch apart, p and q are different colors. (This is open: I was unable to do this in χ minutes.)

Alex uses the symbol χ for this quantity throughout the book; hence we will use the symbol χ for this quantity throughout the review.

The problem of determining χ is called the *Chromatic Number of the Plane Problem* and is abbreviated *CNP*. Alex told me *CNP is the most important problem in all of mathematics.* I think his point was that its important to work on problems that can be explained to the layperson and that he was using this as an illustration. Or maybe he really does think so. I hope he does—the world needs idealists.

After seeing Alex's talk I asked my colleague Clyde Kruskal what happens if only a subset of the plane is colored. For example, what is the largest square that can be 2-colored? 3-colored? Clyde then obtained full characterizations of 2 and 3-colorings for rectangles and regular polygons [**?**]. The paper contains the following marvelous result: an $s \times s$ square is 3-colorable iff $s \leq 8/\sqrt{65}$. (Alex plans to put this result into the next edition.)

After talking to Alex I very much looked forward to his book. I first got my hands on it at the SODA (Symposium on Discrete Algorithms) conference of 2009. The Springer-Verlag book vendor let me read parts of it during the coffee break. I later got a copy and read the whole thing.

2 What Kind of Book is this?

When I first read the book I noticed something odd. The first sentence is *I recall April of 1970.* **Most of the book is written in the first person, like a memoir or autobiography!** The only parts that are not written in first person is when someone else is doing the talking.

In Alex's honor my review is written in his style.

Ordinary math books are not written in the first person; however, this is no ordinary math book! I pity the Library of Congress person who has to classify it. This book contains much math of interest and pointers to more math of interest. All of it has to do with coloring: Coloring the plane (Alex's favorite problem), coloring a graph (e.g., the four color theorem), and of course Ramsey Theory. However, the book also has biographies of the people involved and scholarly discussions of who-conjectured-what-when and who-proved-what-when. When I took Calculus the textbook had a 120-*word* passage about the life of Newton. This book has a 120-*page* passage about the life of van der Waerden.

Is this a math book? YES. Is this a book on history of Math? YES. Is this a personal memoir? YES in that the book explicitly tells us of his interactions with other mathematicians, and implicitly tells us of his love for these type of problems.

Usually I save my opinion of the book for the end. For this book, I can't wait:

This is a Fantastic Book! Go buy it Now!

3 Summary

The book is in eleven parts, each one of which has several chapters. There are 49 chapters in all. While I will pick out results and facts from chapters to discuss, the actual book contains far more in it then I can summarize in my review. I am amazed Alex fit it all into 600 pages.

Part I: Merry-Go-Round

Imagine that you are a judge in a Math Olympiad and you find out, a day before the exam, that the sample solution to one of the problems is wrong! Alex does not need to imagine it since it really happened to him. This Part tells the full story. Not to worry— it has a happy ending. One of the judges found a proper solution before the contest. But Alex does not stop there. Alex then discusses alternate solutions, the history of the problem, and the fact that several first class mathematicians had worked on it over the years.

Part II: Colored Plane

This part is on the CNP discussed above. The part of the book, together with other parts, contain most of what is known about CNP.

The following concept is used throughout the book so I state it here, and use it throughout the review: A *Unit Distance Graph* is a graph obtained by taking a set of points in the plane and connecting two of them if they are an inch apart.

This part introduces CNP and gives history, context, and variants. Here are some things he talks about: (1) CNP was posed by Edward Nelson in 1950. Many references get this wrong. (2) De Bruijn and Erdös (1951) proved that, for any graph G, G is k-colorable iff every finite subset of G is k-colorable. Hence there is a finite number of points in the plane such that χ is the chromatic number of the unit distance graph they form. This proof uses the Axiom of Choice. Today this would be considered a standard compactness result. (3) Variant (by Erdös): Assume the plane is colored. A color RED *realizes a set* $S \subseteq R$ if, for all $x \in S$, there are two points p, q both colored RED such that $d(p, q) = x$. Say you want to color the plane so that no color realizes R. How many colors do you need? It is easy to show that you need at least 3. It is known to be between 4 and 6.

After reading this chapter I believed that CNP *is* the most important problem in mathematics. Then that feeling passed.

Part III: Coloring Graphs

This part begins with standard material on graph coloring. Alex then talks about CNP again! The standard proof that $\chi \geq 4$ involves a unit distance graph on 7 points that cannot be 3-colored. That graph has triangles. What if we could not use triangles? The *girth of a graph* is the length of its shortest cycle.

Erdös made the following conjecture: *For all* $g \geq 3$ *there is a unit distance graph of girth* g *that is not 3-colorable.* I looked ahead: the conjecture was proven by Paul O'Donnell (explained in Part IX). I found the journey described in Part IX fascinating.

Alex tells us about unit distance graphs that have high girth, chromatic number 4, and not that many points. I'll mention one: there is a 45 vertex unit distance graph with girth 5 that is not 3-colorable.

Part IV: Coloring Maps

This chapter is mostly about the four-color theorem. This chapter corrected some of my misconceptions. I thought Heawood made the conjecture (WRONG—it was Francis Guthrie). I thought Kempe's incorrect proof of the 4-color theorem lead to the 5-color theorem (RIGHT with a caveat—Heawood should get joint credit for the 5-color theorem). I thought the error was not discovered for a while because nobody was working on it (WRONG— William E. Story, Alfred Bray Kempe, and Frederick Guthrie Tait all simplified Kempe's "proof" of the 4-color theorem). I thought that since Kempe's proof lead to the 5-color theorem, Kempe made an important contribution (RIGHT). I thought that this positive view of Kempe was the accepted view (WRONG— Alex tracks down many (all?) of the negative comments about Kempe in the literature in order to refute them). The math and history are both fascinating and intertwine nicely.

Recall that the four-color theorem was eventually proved in a way that made extensive use of a computer program. It was the first theorem of interest proven this way. At the time this was controversial. Was it really a proof? This part has a chapter on the controversy. This chapter is excellent in that it has all of the debate in one place and, since time has passed, has far more prospective then the original debates.

Much to my surprise the following facts are not in the book. In 1890 Percy Heawood proved that, for all $g \geq 1$, a graph of genus g can be colored with $\leq \left\lfloor \frac{7+\sqrt{1+48g}}{2} \right\rfloor$ colors. Heawood conjectured that this is optimal. This was proven by Ringel and Young in 1968 [**?**]. This proof did not use a computer program. The fact that the $g = 0$ case (planar graphs) is so much different than the $g \geq 1$ case is interesting. For the case of $g = 1$ it is easy to construct a graph of genus 1 that requires 7 colors. I suspect Heawood knew this, though I was unable to confirm this using Google. If Alex Soifer was researching this issue then he would go to England and look over Heawood's personal papers to confirm.

Part V: Colored Graphs

This part has three chapters in it: (1) A mini biography of Erdös and Alex's interactions with him, (2) a history of the De Bruijn Theorem and its impact (and a proof), and (3) material on edge colorings of graphs: Ramsey's Theorem, Folkman's Theorem, and a table of all known Ramsey numbers, are all included.

All of this information is interesting. The title of the Part comes from the fact that all the chapters are about properties of colored graphs.

Part VI: The Ramsey Principle

This part has the infinite Ramsey Theorem, some applications of both the infinite and finite Ramsey Theorem, and a biography of Ramsey (it's short as, alas, so was Ramsey's life). One of the applications has taken on a life of its own: the *Happy Ending Problem.* Alex gives the full story of this problem plus where the major participants in its origin ended up.

This part also has some of the principle underlying Ramsey Theory. Almost all theorems in Ramsey Theory involve coloring some structure and finding some regular monochromatic substructure. Hence they are all about finding **order**. Alex includes the famous quote by Motzkin, *Complete disorder is impossible* as the rallying cry for Ramsey Theory.

Part VII: Colored Integers: Ramsey Theory before Ramsey and its AfterMath

The first chapter of this part discusses Hilbert's Cube Lemma, which may be the first Ramseyian result ever. Hilbert used it as a lemma in his work on irreducible polynomials, but never returned to studying Ramseyian problems.

The second chapter of this part discusses Schur and Rado's work on equations. Schur proved the following:

(Schur's Theorem) For any finite coloring of N there exists x, y, z that are the same color such that $x + y = z$.

Rado generalized this to other types of linear equations. The chapter discusses many generalizations beyond Rado's.

The third chapter is a reprint of an article by van der Waerden about how the theorem that bares his name was discovered. I state it in a particular way so that it is similar to the polynomial van der Waerden theorem stated later.

(Schur-Baudet-van der Waerden's Theorem[2]*) For all k, for all c, there exists W such that, for all c-colorings of $\{1,\ldots,W\}$ there exists a, d such that*

$$a, a+d, \ldots, a+kd \text{ are all the same color.}$$

The discovery was a group collaboration. I noticed (1) from this discussion one can easily construct the proof of van der Waerden's theorem (the book does not have a proof presented in the standard way), and (2) Emil Artin should have been a co-author.

The book also mentions the following generalization of the Schur-Baudet-van der Waerden Theorem:

(Canonical VDW's Theorem) Any coloring of the positive integers in infinitely many colors contains arbitrarily long monochromatic or representative (all different colors) arithmetic sequences.

The proof is attributed to Erdös-Graham. I've seen the reference and it claims that this follows *easily* from Szemerédi's theorem. Alex also claims that it follows from Szemerédi's theorem but does not offer an opinion on the difficulty. I personally would not call the proof obvious. The only place to read the implication of this from Szemerédi's theorem is a comment on my complexity blog [**?**]. Hans Jürgen Rödl and Vojtech Prömel [**?**] have a purely combinatorial proof. Alex did not include the fact that there is a purely combinatorial proof! I was surprised— there is actually something about coloring that I knew that he didn't! (I have since told him.) This only happened one other time in the book.

The first three chapters had some history in them. The fourth chapter is entirely history. Van der Waerden's paper is titled (when translated) *On a conjecture of Baudet.* Alex spends 21 pages arguing that the conjecture is actually jointly and independently Baudet's and Schur's. I have to ask myself *do I care?* Alex made me care! Alex points out that this kind of thinking was entirely new in mathematics and hence it is important to know who had these ideas. Alex also insists on calling *van der Waerden's theorem* by the name *the Baudet-Schur-van der Waerden Theorem.*

The fifth chapter discusses variants and generalizations of the Schur-Baudet-van der Waerden Theorem. I was particularly eager to see if he included my favorite— the polynomial van der Waerden theorem first proven by Bergelson-Leibman [**?**] using ergodic theory.

(Polynomial VDW theorem) For all $p_1, \ldots, p_k \in Z[x]$ such that $p_1(0) = \cdots = p_k(0) = 0$, for all c, there exists W such that, for all c-colorings of $\{1,\ldots,W\}$ there exists a, d such that

$$a, a+p_1(d), a+p_2(d) \ldots, a+p_k(d) \text{ are all the same color.}$$

Alex included it! Alex did not include the fact that there is a purely combinatorial proof by Walters [**?**]. I was surprised— this is the second fact about coloring that I knew that he didn't! (I have since told him.) The chapter also includes discussion of Szemerédi's theorem (no proof, which is appropriate). The chapter *did not* have Gallai's theorem (the multidimensional VDW theorem), though that is in a later chapter. In a normal math book I would wonder why Gallai's theorem was not in this chapter. This book is already so unusual, that this is not that striking.

[2] Alex names theorems after both those who conjectured them and who proved them. We follow his style.

The sixth, seventh, eight and ninth chapters are actually the first ones I read. That is because they are all history which one can read during a sequence of SODA conference coffee break. These chapters are an extremely scholarly treatment of the life of van der Waerden. Why does van der Waerden get this lengthy treatment and others did not? (1) There are no biographies of van der Waerden. (2) He lived a long and interesting life that raises many moral questions. The key one is is his behavior when he lived in Nazi Germany. Given my first meeting with Alex I expected the portrayal to be negative. However, Alex intentionally lays out the facts and lets you decide. Unlike Mathematics it is hard to come to a definite conclusion.

Part VIII: Colored Polygons: Euclidean Ramsey Theory

Alex begins by giving us a fascinating tour of problems of the following sort: Let T be a triangle and $c \in N$. Consider the following two statements: (1) For any c-coloring of the plane there exists a triangle *congruent* to T such that all the vertices are the same color. (2) For any c-coloring of the plane there exists a triangle *similar* to T such that all the vertices are the same color. Euclidean Ramsey Theory tries to see for which triangles (and other shapes) such statements are true.

For triangles and congruence even for 2-colorings this is open; however, the following is known:

1. For all right triangles T, for all 2-colorings of the plane, there exists a right triangle congruent to T that has all three vertices the same color.

2. For all trapezoids $TRAP$, for all finite colorings of the plane, there exists a trapezoid congruent to $TRAP$ with all four vertices the same color.

3. Let T be the triangle that has short side of length 1 and angles in the ratio 1:2:3. (1) For any 2-coloring of the plane there exists a triangle congruent to T with all three vertices the same color. (2) For any 3-coloring of the plane there exists a triangle congruent to T with either all three vertices the same color or all three vertices different colors.

4. For all triangles T, for all finite colorings of the plane, there exists a right triangle similar to T that has all three vertices the same color. (This follows from Gallai's theorem which is discussed next.)

Alex then turns to Gallai's theorem. Imagine that you 2-color the *lattice points* of the plane instead of the entire plane. What monochromatic shapes are you guaranteed? Gallai's Theorem (which is sometimes called the Gallai-Witt theorem) gives a partial answer for coloring lattice points of R^n. I present the two-dimensional version.

Let $A \subseteq R^2$. A set $B \subseteq R^2$ is *homothetic to A* if there is an affine function $f : R^2 \rightarrow R^2$ such that, when restricted to A, is a bijection onto B. For example, if $A = \{(0,0),(0,1),(1,0),(1,1)\}$ then any square with sides parallel to the x-axis and y-axis axis is homothetic to A.

(Gallai's Theorem) For all $A \subseteq Z^2$, for all $c \in N$, for all c-colorings of $Z^2 \times Z^2$, there exists a monochromatic set $B \subseteq Z^2$ that is homothetic to A.

Alex gives several proofs of this theorem on his way to establishing authorship and biography. Within 20 pages you encounter (1) math of interest, (2) scholarly discussion of who deserves credit: Gallai alone, not Witt or Garsia. (Who is Garsia? Alex tells us.) (3) biographical information about Witt being a member of the Nazi party and Hasse being overtly anti-Jewish, anti-black, and anti-Polish. How did Hasse get into this story? Hasse was one of Witt's teachers. This chapter is a microcosm of the entire book— a breathtaking whirlwind of math and history.

Part IX: Colored Integers in Service of Chromatic Number of the Plane

I am often asked if there are any applications of the Schur-Baudet-van der Waerden Theorem or the Polynomial van der Waerden Theorem. I have a website of applications of Ramsey Theory [**?**]; however, most of the applications are of Ramsey's Theorem, not the Schur-Baudet-van der Waerden Theorem.

I was delighted and surprised to read the following theorems by Paul O'Donnell. (1) Using the Schur-Baudet-van der Waerden Theorem a unit distance graph of girth 9 and chromatic number 4 is constructed. (2) Using the polynomial van der Waerden Theorem and the Mordell-Falting theorem a unit distance graph of girth 12 and chromatic number 4 is constructed. The proofs are in this book and are well written. I have already gone over them carefully and presented them to my van der Waerden gang.[3]

Later Paul O'Donnell proved Erdös's conjecture on unit chromatic graphs of high girth without any of these tools. That should make me happy but it makes me sad. Even so, I still count the results (1) and (2) above as applications.

Part X: Predicting the Future

One reason that CNP is so hard is that it has to deal with *any* coloring of the plane. Recall that in this review's summary of Part II, I mentioned the following result: the CNP problem is equivalent to CNP on all finite sets of points, and the proof of this uses the Axiom of Choice (AC). What if AC was not available? Most of this chapter is about the fascinating results of Shelah and Soifer on what could happen if AC is replaced by other (reasonable) axioms. They do not get out a statement about CNP but they do get statements about related problems.

Part XI: Farewell to the Reader

The last paragraph summarizes this 2-page chapter and Alex's attitude:
Thank you for inviting my book into your home and holding it in your hands. Your feedback, problems, conjectures, and solutions will always be welcome in my home. Who knows, maybe they will inspire a new edition in the future and we will meet again.

4 Opinion

Who *could* read this book? The upward closure of the union of the following people: (1) an excellent high school student, (2) a very good college math major, (3) a good grad student in math or math-related field, (4) a fair PhD in combinatorics, or (5) a bad math professor.

Who *should* read this book? Anyone who is interested in math or history of math. This book has plenty of both. If you are interested in math then this book will make you interested in history of math. If you are interested in history of math then this book will make you interested in math. Any researcher in either mathematics or the history of mathematics, no matter how sophisticated, will find many interesting things they did not know.

[3] My van der Waerden gang is a factorial of bright high school students who are doing projects with me in Ramsey Theory.

References

[1] V. Bergelson and A. Leibman. Polynomial extensions of van der Waerden's and Szemerédi's theorems. *Journal of the American Mathematical Society*, pages 725–753, 1996. http://www.math.ohio-state.edu/~vitaly/ or http://www.cs.umd.edu/~gasarch/vdw/vdw.html.

[2] A. Drucker. Proof that canonical vdw theorem follows from Szemeredi's theorem, 2009. A comment on the blog http://weblog.fortnow.com/2009/02/seeking-non-elementary-proof-of.html.

[3] W. Gasarch. Applications of Ramsey Theory (website of papers). www.cs.umd.edu/~gasarch/ramsey/ramsey.html.

[4] C. Kruskal. The chromatic number of the plane: the bounded case. *Journal of Computer and System Sciences*, 74:598–627, 2008. www.cs.umd.edu/~kruskal/papers/papers.html.

[5] H. J. Prömel and V. Prömel. An elementary proof of the canonizing version of Gallai-Witt's theorem. *JCTA*, 42:144–149, 1986. http://www.cs.umd.edu/~gasarch/vdw/vdw.html.

[6] G. Ringel and J. W. T. Young. Solution of the Heawood map colouring problem. *Proc. of the National Academy of Science (USA)*, 60:438–445, 1967.

[7] M. Walters. Combinatorial proofs of the polynomial van der Waerden theorem and the polynomial Hales-Jewett theorem. *Journal of the London Mathematical Society*, 61:1–12, 2000. http://jlms.oxfordjournals.org/cgi/reprint/61/1/1 or http://jlms.oxfordjournals.org/ or or http://www.cs.umd.edu/~gasarch/vdw/vdw.html.