A Math Problem and its Solution

Problem

Find the coefficient of x^{100} in the Taylor expansion of

$$\frac{1}{x^{41}-x^{40}-x^{36}+x^{35}-x^{31}+x^{30}+x^{26}-x^{25}-x^{16}+x^{15}+x^{11}-x^{10}+x^{6}-x^{5}-x+1}.$$

The Solution Note that the denominator factor into $(x-1)(x^5-1)(x^{10}-1)(x^{25}-1)$. We rewrite the problem as finding the coefficient of x^{100} in

$$\frac{1}{1-x}\frac{1}{1-x^5}\frac{1}{1-x^{10}}\frac{1}{1-x^{25}}$$

$$= (1+x+x^2+\cdots)(1+x^5+x^{10}+\cdots)(1+x^{10}+x20+\cdots)(1+x^{25}+x^{50}+\cdots).$$

The coefficient of x^n is the number of ways to make change for n cents using pennies, nickels, dimes, and quarters. So our problem is how many ways can you make change for a dollar.

Let a_n be the number of ways to make change with pennies. So

$$(\forall n \ge 0)[a_n = 1].$$

Let b_n be the number of ways to make change with pennies and nickel. So

$$(\forall n \ge 5)[b_n = b_{n-5} + a_n]$$

It is easy to show that

$$(\forall n \geq 0)(\forall i \in \{0,1,2,3,4\})[b_{5n+i} = n+1].$$

Let c_n be the number of ways to make change with pennies, nickels, and dimes. So

$$(\forall n \geq 10)[c_n = c_{n-10} + b_n]$$

Let d_n be the number of ways to make change with pennies, nickels, dimes, and quarters. So

$$(\forall n \geq 10)[d_n = d_{n-25} + c_n]$$

We need to compute d_{100}. We use the exact formula for b_n and the recurrences for c_n and d_n.

$d_{100} = c_{100} + c_{75} + c_{50} + c_{25} + c_0$

$c_0 = 1$

$c_{25} = b_{25} + b_{15} + b_5 = 6 + 4 + 2 = 12$

$c_{50} = b_{50} + b_{40} + b_{30} + b_{20} + b_{10} + b_0 = 11 + 9 + 7 + 5 + 3 + 1 = 36$

$c_{75} = b_{75} + b_{65} + b_{55} + b_{45} + b_{35} + c_{25} = 16 + 14 + 12 + 10 + 8 + 12 = 72$

$c_{100} = b_{100} + b_{90} + b_{80} + b_{70} + b_{60} + c_{50} = 21 + 19 + 17 + 15 + 13 + 36 = 121$

Hence

$d_{100} = 1 + 12 + 36 + 72 + 121 = 242.$

Hence the coefficient of x^{100} is 242.