UNIT-DISTANCE GRAPHS IN RATIONAL \( n \)-SPACES

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Let \( U_n \) be the infinite graph with \( n \)-dimensional rational space \( \mathbb{Q}^n \) as vertex set and two vertices joined by an edge if and only if the distance between them is exactly 1. The connectedness and clique numbers of the graphs \( U_n \) are discussed.

1. Introduction and definitions

Let \( R^n \) and \( \mathbb{Q}^n \) denote real and rational \( n \)-space, equipped with the usual Euclidean metric. Let \( G_n \) denote the infinite graph whose vertices are the points of \( R^n \), two vertices adjacent if and only if the distance between them is exactly 1. It is easy to see that \( G_n \) is connected for \( n \geq 2 \) and the maximum number of points in \( R^n \) that are pairwise unit distance apart (the clique number of \( G_n \)) is \( n + 1 \) for \( n \geq 1 \). However, the chromatic number of \( G_n \) is so far unknown for \( n \geq 2 \) [1].

Let \( U_n \) be the subgraph of \( G_n \) induced by those vertices that are in \( \mathbb{Q}^n \). In Section 2 we shall prove that \( U_n \) is connected if and only if \( n \geq 5 \). In Section 3 we shall determine the clique number \( \omega(n) \) of \( U_n \). For even \( n \), \( \omega(n) \) is \( n + 1 \) or \( n \) according as \( n + 1 \) is or is not a perfect square. For odd \( n \), if the diophantine equation \( nx^2 - 2(n - 1)y^2 = z^2 \) has an integer solution \((x, y, z)\) with \( x \neq 0 \), then \( \omega(n) = n + 1 \) or \( n \) according as \( \frac{1}{2}(n + 1) \) is or is not a perfect square; otherwise, \( \omega(n) = n - 1 \).

2. The connectedness of \( U_n \)

In this section we shall first prove that \( U_1, U_2, U_3, \) and \( U_4 \) are all disconnected and prove that \( U_n \) is connected for \( n \geq 5 \).

Lemma 1. There is no path in \( U_4 \) connecting the origin \((0, 0, 0, 0)\) to \((\frac{1}{4}, 0, 0, 0)\).

Proof. Suppose there is. Then, equivalently, there are finitely many points on the unit sphere in \( \mathbb{Q}^4 \) whose sum is \((\frac{1}{4}, 0, 0, 0)\). Let \((a_1/b, a_2/b, a_3/b, a_4/b)\) be such a point, where \( a_1, a_2, a_3, a_4, \) and \( b \) have no common factor and

\[
a_1^2 + a_2^2 + a_3^2 + a_4^2 = b^2. \tag{1}
\]
If $b$ is divisible by 4, then at least one of $a_1, a_2, a_3, a_4$ is odd, and so the left-hand side of (1) is not divisible by 8 whereas the right-hand side is. (Recall that the only squares modulo 8 are 0, 1, and 4.) Thus $b$ is either odd or twice an odd integer. But the sum of a finite number of fractions with denominators of this form cannot be equal to $\frac{1}{4}$. This completes the proof of the lemma. \qed

**Theorem 2.** The graphs $U_1, U_2, U_3,$ and $U_4$ are all disconnected.

**Proof.** This follows immediately from Lemma 1, since there are obvious subgraphs of $U_4$ that contain the points $(0, 0, 0, 0)$ and $(\frac{1}{4}, 0, 0, 0)$ and are isomorphic to $U_1$, $U_2$, and $U_3$, respectively. \qed

**Theorem 3.** The graph $U_n$ is connected for $n \geq 5$.

**Proof.** First note that if there exist two paths in $U_n$, one connecting 0 to $x$ and the other connecting 0 to $y$, then there exists a path from 0 to $x + y$ in $U_n$. With this observation, it suffices to show that there is a path from 0 to $(0, 0, \ldots, 0, 1/N, 0, \ldots, 0)$ in $U_n$ for every non-zero integer $N$ with $1/N$ in the $i$th coordinate for $i = 1, 2, \ldots, n$. Consider the integer $4N^2 - 1$. Since it is positive it can be written as a sum of four squares by Lagrange’s Four Square Theorem. Hence, $4N^2 - 1 = a^2 + b^2 + c^2 + d^2$ for some integers $a, b, c, \text{ and } d$, or, equivalently,

$$1 = \left(\frac{1}{2N}\right)^2 + \left(\frac{a}{2N}\right)^2 + \left(\frac{b}{2N}\right)^2 + \left(\frac{c}{2N}\right)^2 + \left(\frac{d}{2N}\right)^2. \quad (2)$$

So, there are edges in $U_n$ joining 0 and

$$\left(\frac{1}{2N}, \pm \frac{a}{2N}, \pm \frac{b}{2N}, \pm \frac{c}{2N}, \pm \frac{d}{2N}, 0, 0, \ldots, 0\right).$$

This shows that there is a path of length 2 in $U_n$ connecting 0 to $(1/N, 0, 0, \ldots, 0)$. By repeating the above with $1/2N$ in the $i$th coordinate, the desired path is obtained. This completes the proof of the theorem. \qed

3. The clique number of $U_n$

A set of points will be called unidistant if they are pairwise unit distance apart. Let $\omega(n)$ denote the maximum number of unidistant points in $Q^n$ (the clique number of $U_n$). We may remark that any unidistant set can be translated so that the translated unidistant set contains 0. In this section, we first find bounds for $\omega(n)$ and then evaluate $\omega(n)$.

**Lemma 4.** $\omega(n) \leq n + 1$. 
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Proof. Let \( \{0, y_1, y_2, \ldots, y_r\} \) be a unidistant set in \( \mathbb{Q}^n \). Let \( A \) be the \( r \times n \) matrix whose rows are \( y_1, y_2, \ldots, y_r \). Now the \( r \times r \) matrix \( AA^T \) has 1's on the principal diagonal and \( \frac{1}{2} \) everywhere else. \( AA^T \) is a non-singular matrix and so,

\[ r = \text{rank}(AA^T) = \text{rank}(A) \leq n. \]

From this it follows immediately that \( \omega(n) \leq n + 1 \). This completes the proof of the lemma. \( \Box \)

Lemma 5. If \( n \geq 4 \), then \( \omega(n) = n \) if \( n \) is even and \( \omega(n) = n + 1 \) if \( n \) is odd.

Proof. If \( n \) is even, define a set \( S_n \) of \( n \) unidistant points as follows:

\[
\begin{align*}
x_1 &= 0 \\
x_2 &= (1, 0, 0, \ldots, 0) \\
x_3 &= (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0) \\
x_4 &= (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0, \ldots, 0) \\
x_5 &= (\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, 0, \ldots, 0) \\
x_6 &= (\frac{1}{2}, \frac{1}{2}, 0, 0, -\frac{1}{2}, 0, \ldots, 0) \\
&& \vdots \\
x_{n-1} &= (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0, \frac{1}{2}, \frac{1}{2}) \\
x_n &= (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0, \frac{1}{2}, -\frac{1}{2})
\end{align*}
\]

If \( n \) is odd, define a set \( T_n \) of \( n - 1 \) unidistant points by adding an extra coordinate zero to the end of each vector in \( S_{n-1} \). \( \Box \)

Theorem 6. \( \omega(n) = n + 1 \) if and only if a set of \( n \) unidistant points exist in \( \mathbb{Q}^n \) and \( (n + 1)/2^n \) is a rational square.

Proof. If \( \omega(n) = n + 1 \), then with no loss of generality let \( \{0, x_1, \ldots, x_n\} \) be a set of the \( n + 1 \) unidistant points in \( \mathbb{Q}^n \). Let \( A \) be the \( n \times n \) matrix having \( x_1, x_2, \ldots, x_n \) as its rows. It is clear that \( \det(A) \) (the determinant of \( A \)) is a rational number. No < \( \det(4A^T) = (n + 1)/2^n = \) square of \( \det(A) \), thus showing that \( (n + 1)/2^n \) is a rational square.

Suppose \( (n + 1)/2^n \) is a rational square and \( \{0, x_1, \ldots, x_{n-1}\} \) is a unidistant set of \( n \) points. We will construct a point \( x_n \) so that \( \{0, x_1, \ldots, x_n\} \) is a unidistant set in \( \mathbb{Q}^n \). Consider the \( (n - 1) \times n \) matrix \( B \) having \( x_1, \ldots, x_{n-1} \) as its rows. Let \( B_i \) be the \( (n - 1) \times (n - 1) \) matrix obtained from \( B \) by deleting its \( i \)th column, and let \( a_i = (-1)^{i+1} \det(B_i) \), for \( i = 1, 2, \ldots, n \). Defining a vector \( x = (a_1, a_2, \ldots, a_n) \), we observe that it has the following properties;

1. \( x \) is in \( \mathbb{Q}^n \).
2. \( x \) is orthogonal to \( x_1, x_2, \ldots, x_{n-1} \) (follows from construction),
3. \( \|x\|^2 = \det(BB^T) = n/2^{n-1} \) (easily verified and also a consequence of the Cauchy–Binet Theorem).
Define a vector \( x_n = kx + c \), where
\[
c = \frac{1}{n} (x_1 + x_2 + \cdots + x_{n-1})
\]
and
\[
k = \frac{2^{n-1}}{n} \sqrt{\frac{n + 1}{2^n}}.
\]
The vector \( x_n \) is in \( Q^n \) since \( k \) is a rational number. From properties (2) and (3) above, it follows that
\[
\|x_n\|^2 = k^2 \|x\|^2 + 2kx \cdot c + \|c\|^2
\]
\[
= \frac{2^{n-2} n + 1}{n^2} \cdot \frac{n}{2^n} + \frac{n - 1}{n^2} \left( n - 1 + \frac{(n-1)(n-2)}{2} \right)
\]
\[
= \frac{n + 1}{2n} + \frac{n - 1}{2n} = 1
\]
and
\[
\|x_n - x_i\|^2 = \|x_n\|^2 - 2x_n \cdot x_i + \|x_i\|^2
\]
\[
= 1 - \frac{2}{n} \left( 1 + \frac{n-2}{2} \right) + 1 = 1, \quad \text{for } i = 1, 2, \ldots, n - 1. \quad (4)
\]
This completes the proof. \( \square \)

**Theorem 7.** If \( n \) is even, then \( \omega(n) = n + 1 \) if \( n + 1 \) is a perfect square and \( \omega(n) = n \) otherwise.

**Proof.** If \( n \geq 4 \), this follows immediately from Lemma 5 and Theorem 6. If \( n = 2 \), the result is a simple exercise. In fact, Woodall [4] shows that \( U_2 \) is two-colorable (bipartite). \( \square \)

In what follows, we shall need the following theorem:

**Theorem (Hall and Ryser [2]).** Let \( A \) be a non-singular \( n \times n \) matrix with entries from a field of characteristic \( \neq 2 \), and suppose that \( AA^T = D_1 \oplus D_2 \), the direct sum of two square matrices \( D_1 \) and \( D_2 \) of orders \( r \) and \( s \) respectively (\( r + s = n \)). Let \( M \) be an arbitrary \( r \times n \) matrix such that \( MM^T = D_1 \). Then there exists an \( n \times n \) matrix \( Z \) having \( M \) as its first \( r \) rows such that \( ZZ^T = D_1 \oplus D_2 \).

**Lemma 8.** Let \( U \) and \( V \) be two unidistant sets of \( n - 1 \) points in \( Q^n \). Then there is a rational orthogonal transformation (preserving distances and inner products) that maps \( U \) onto \( V \). In particular, there is a point \( u \) in \( Q^n \) that is unidistant from all points in \( U \) if and only if there is a point \( v \) in \( Q^n \) that is unidistant from all points in \( V \).
Proof. There is no loss of generality in supposing that $\emptyset$ is in both $U$ and $V$, so that we can write

$$U = \{\emptyset, u_1, \ldots, u_{n-2}\} \quad \text{and} \quad V = \{\emptyset, v_1, \ldots, v_{n-2}\}.$$ 

Let $u_{n-1}$ and $u_n$ be independent vectors in $Q^n$ that are orthogonal to all the vectors in $U$. Let $A$ be the $n \times n$ matrix with rows $u_1, u_2, \ldots, u_n$ and let $M$ be the $(n-2) \times n$ matrix with rows $v_1, v_2, \ldots, v_{n-2}$. Then $A$ is non-singular, $AA^T = D_1 \oplus D_2$ and $MM^T = D_1$, where $D_1$ is a square matrix of order $n-2$ with 1's on the principal diagonal and $\frac{1}{2}$ everywhere else, and $D_2$ is a non-singular $2 \times 2$ matrix. By Hall and Ryser's theorem, there exists an $n \times n$ matrix $Z$ having $M$ as its first $n-2$ rows such that $ZZ^T = D_1 \oplus D_2$. Let $L = Z^{-1}A$. Then $L$ is a rational matrix such that $v_iL = u_i$, for $i = 1, 2, \ldots, n-2$. Moreover, $L$ is an orthogonal matrix, because $(Z^T)^{-1}Z^{-1}AA^T = I$ and so $LL^T = Z^{-1}AA^T(Z^{-1})^T = I$. This completes the proof of Lemma 8. \[\square\]

**Theorem 9.** Let $n$ be an odd integer $\geq 5$. If the diophantine equation

$$nx^2 - 2(n-1)y^2 = z^2 \quad (5)$$

has an integer solution $(x, y, z)$ with $x \neq 0$, then $\omega(n) = n+1$ or $n$ according as $\frac{1}{2}(n+1)$ is or is not a perfect square; otherwise $\omega(n) = n - 1$.

**Proof.** In view of Theorem 6, it suffices to prove that $\omega(n) \geq n$ if and only if (5) has an integer solution with $x \neq 0$. By Lemma 8, $\omega(n) \geq n$ if and only if there is a point $x$ in $Q^n$ that is unidistant from all the $n-1$ points in the set $T_n$ of Lemma 5. Let

$$x = (t_1, s_1, t_2, s_2, \ldots, t_m, s_m, r)$$

be such a point, where $m = \frac{1}{2}(n-1)$. It follows immediately that $t_1 = \frac{1}{2}$, $s_2 = s_3 = \cdots = s_m = 0$, $t_2 = t_3 = \cdots = t_m = \frac{1}{2} - s_1$ and $s_1^2 + (m-1)(\frac{1}{2} - s_1)^2 + r^2 = \frac{3}{4}$. Solving for $s_1$ in terms of $r$,

$$s_1 = \frac{m-1 \pm \sqrt{n-4mr^2}}{2m}. \quad (6)$$

Thus there exists a point $x$ in $Q^n$ as required if and only if there exists a rational number $r = y/x$ such that $n - 4mr^2$ is a rational square, say $(z/x)^2$; that is, if and only if eq. (5) has an integer solution with $x \neq 0$. This completes the proof of Theorem 9. \[\square\]

The above theorem is also true for $n = 1$ and $n = 3$. For $n = 3$, the result is a simple exercise. The chromatic number of $U_3$ is 2. Robertson [3] has shown that the chromatic number of $U_4$ is 4. These results will be reported in a separate paper dealing mainly with the coloring of graphs $U_n$. 

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References