REU Project on Van der Waerden Games

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1 Introduction

What is Van Der Waerden's theorem? I begin by giving an example.

BILL: Clyde, pick a number.

CLYDE: Okay, 19.

BILL: Great! There is a number N such that no matter how you 2-color the numbers $\{1, 2, \ldots, N\}$ there will exist 19 numbers, *equally spaced(!)* that are the same color.

CLYDE: Really?

BILL: Yes!

CLYDE: what if I had picked 119 or 10^{10}?

BILL: For any of those numbers YES there is an N that works.

CLYDE: what if I 3-colored instead of 2-colored?

BILL: The theorem still holds.

CLYDE: Okay. Can you state all of this formally?

BILL: I will state it a few different ways.

Def 1.1 An *arithmetic progression of length k* is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + (k - 1)d$$

Note that the elements are equally spaced. We appreviate this as a k-AP.

Example: 4,10,16,22,28 is an arithmetic progression of length 5.

Theorem 1.2 *Van der Waerden's Theorem: For all c, k, there exists W such that, for every c-coloring of $\{1, \ldots, W\}$ there exists a monochromatic k-AP.*

We can restate this as:

Theorem 1.3 *Van der Waerden's Theorem: For all c, k, there exists W such that, for every c-coloring $COL : \{1, \ldots, W\} \rightarrow \{1, \ldots, c\}$ there exists a, d such that*

$$COL(a) = COL(a + d) = \cdots = COL(a + (k - 1)d.$$

Def 1.4 If $k, c \in \mathsf{N}$ then $W(k, c)$ is the least number W such that for every c-coloring of $\{1, \ldots, W\}$ there exists a monochromatic k-AP. Such a W is guaranteed to exist by VDW's theeorem These are called *the van der Waerden Numbers* or *the VDW numbers*.

Very few VDW numbers are known.

2 View VDW's Theorem as Game

It is known that $W(3, 2) = 9$. That is, no matter how one 2-colors $\{1, \ldots, 9\}$ there will be a mono 3-AP.

Consider the following game:

1. The two players are named RED (who goes first) and BLUE (who goes second).

2. The players alternate coloring the numbers $\{1, \ldots, 9\}$.

3. The first player to have a mono 3-AP in his color wins.

Here is the key: by VDW's theorem, no matter how the two players play (even if they play badly) *one of them has to win!*.

What if they play well?

Def 2.1 Let $k \in \mathsf{N}$. The *VDW game with parameter k, n* (denoted $VGAME(k, n)$) proceeds as follows:

1. The two players are named RED (who goes first) and BLUE (who goes second).
2. The players alternate coloring the numbers $\{1, \ldots, n\}$.
3. The first player to have a mono k-AP in his color wins.

Def 2.2 If $k \in \mathsf{N}$ then $GW(k)$ is the least number n such that if two players play the $VGAME(k, n)$ perfectly then one of them must win (it will be player I). Note that $GW(k) \leq W(k, 2)$.

3 Our Goal

There are many techniques in AI for game playing. As is well known, there are chess playing programs that beat the world champions. There are programs for Go that do pretty well. The programs for chess and Go are very different. Other games requires other techniques.

In this project we will apply AI game playing techniques to VDW games. One of our goals is to find or a least bound the size of the $GW(k)$. Very little is known about it, however, it may be easier to deal with then $W(k, 2)$.

So the project will proceed as follows:

1. Learn min-max game playing technique. Code it up.
2. Learn alpha-beta pruning. Code it up.
3. Apply both of these to VDW games.
4. Learn monte carlo methods for game playing.
5. Apply that to VDW games.
6. By observing the numbers and the play of the game we may obtain some theorems and proofs (this happened in our REU program in 2014).
7. (Optional) Look at other VDW-type games. See next section.

4 Other VDW-type Games

The following theorems are also known and can be viewed in terms of games

1. For every 2-coloring of the 5×5 grid there will be a mono rectangle (that is, all four corners the same color).
2. There exists a number N such that for every 2-coloring of the $N \times N$ grid there will be a mono square (that is, all four corners the same color).
3. (Informally) Let S be some shape. There exists numbers a, b such that for every 2-coloring of the $a \times b$ grid there will be a mono S.