$W(3,r) \ge e^{A(\ln r)^2}$

Expostion by William Gasarch

On my blog I stated that one could easily get $W(3,r) \ge r^{3/2}$. A commenter, who is unfortuantely anonymous, sketched a proof that $W(3,r) \ge e^{A(\ln r)^2}$ for some constant A. This note is an exposition of the proof. It is not due to me and I doubt it was due to the commenter; I think it is well known.

Behrend [1] showed the following:

Lemma 0.1 *for all n, there is a 3-free subset of $[n]$ of size $\ge ne^{-c\sqrt{\log n}}$ where c is some positive constant.*

Chandra-Furst-Lipton [2] showed the following with an easy probabilistic argument.

Lemma 0.2 *Let $A \subseteq [n]$. There exists r translations of A that cover $[n]$ where $r = O(\frac{n \log n}{|A|})$.*

The following is easily derived Lemma 0.2:

Lemma 0.3 *Let $n \in \mathsf{N}$. Let A be a 3-free subset of $[n]$. For all n there exists a 3-free r-coloring of $[n]$ where $r = O(\frac{n \log n}{|A|})$.*

Combining Lemma 0.1 with Lemma 0.3 we obtain the following:

Theorem 0.4 *For all $n \in \mathsf{N}$ there is a 3-free coloring of $[n]$ that uses r colors where $r = O((\ln n)(e^{c\sqrt{\ln n}}))$*

Proof: Apply Lemma 0.3 to the set A from Lemma 0.3. Note that

$$r = O\left(\frac{n \log n}{ne^{-c\sqrt{\log n}}}\right) = O\left(\frac{\log n}{e^{-c\sqrt{\log n}}}\right) = O((\log n)(e^{c\sqrt{\log n}})) = O((\ln n)(e^{c\sqrt{\ln}}))$$

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So we have that, for all n, there is a 3-free coloring with r colors where r is as above. We want to invert this.

Fix r. Let $n = e^{C(\ln r)^2}$ where C is a constant to be named later. Can $[n]$ be colored with r colors? Need

$$r \geq B(\ln n)(e^{c\sqrt{\ln n}})$$

Note that

$$\ln n = C(\ln r)^2.$$

$$c\sqrt{\ln n} = c\sqrt{C}\ln r = \ln r^{c\sqrt{C}}, \text{ so } e^{c\sqrt{\ln n}} = r^{c\sqrt{C}}$$

So need

$$r \geq BC(\ln r)^2 r^{c\sqrt{C}}$$

Take $C < \frac{1}{c^2}$ to achieve his.

References

[1] F. Behrend. On set of integers which contain no three in arithmetic progression. *Proc. of the National Academy of Science (USA)*, 23:331–332, 1946. `http://www.pnas.org/content/32/12/331.full.pdf+html?sid=aeac9fed-6205-4b25-9f16-7a34a17e6042`.

[2] A. Chandra, M. Furst, and R. Lipton. Multiparty protocols. In *Proceedings of the Fifteenth Annual ACM Symposium on the Theory of Computing,* Boston MA, pages 94–99, 1983. `http://portal.acm.org/citation.cfm?id=808737`.