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From the President

Dear readers of *Mathematics Competitions* journal!

I wish to start my first message as President of WFNMC by expressing thanks to those who attended the Congress in Graz in July 2018 and participated in the discussion of the Constitutional Amendments proposed by Alexander Soifer our Immediate Past President. The approved Amendments establish up to two 4-year consecutive terms limits for certain Officers and Members of the Standing Committees, and a full voting membership on the Executive for Past Presidents of the Federation, who are willing to continue their contributions to the Executive and the Federation.

In Graz we elected the Executive and the Standing Committees of the Federation. I am glad to see new names there and I hope we all will do our best for the future prosperity of the Federation.

As already reported, WFNMC journal Mathematics Competitions has new editor. It is my pleasure to thank the past editor Jaroslav Švrček for his longstanding dedicated work and contributions to make the journal so impressive. At the same time I congratulate the new editor Alexander Soifer and the assistant editor Sergey Dorichenko. I am sure they will continue the increase of the role and influence of *Mathematics Competitions* as a unique journal in this area. I appeal to you, dear colleagues, with the call to help the new editors by submitting interesting materials to the journal.

I realize that the sound foundations of my presidency are built by the past presidents of the Federation: Peter O'Halloran, Blagovest Sendov, Ron Dunkley, Peter Taylor, Petar Kenderov, Maria Falk de Losada, and Alexander Soifer. This is a good start. I hope to continue the directions and the traditions in the development of the Federation.

Finally, let me cite a thought of Alexander Soifer that I like a lot: „The future of the Federation is in our hands“. By saying this I want to stress that I rely on you all to work together for the aims of WFNMC.

With warm regards,
(Kiril Bankov)
President of WFNMC
May 2019

Editor's Page

Dear Competitions enthusiasts,
readers of our *Mathematics Competitions* journal!

Sergei Dorichenko and I are inviting you to submit to our *Mathematics Competitions* your creative essays on a variety of topics related to creating original problems, working with students and teachers, organizing and running mathematics competitions, historical and philosophical views on mathematics and closely related fields, and even your original literary works related to mathematics.

Just be original, creative, and inspirational. Share your ideas, problems, conjectures, and solution with all your colleagues by publishing them here.

We have formalized submission format for establishing uniformity in our journal.

Submission Format

Please, submit your essay to the Editor **Alexander Soifer**.

Format: should be LaTeX, TeX, or Microsoft Word, accompanied by another copy in pdf.

Illustrations: must be inserted at about the correct place of the text of your submission in one of the following formats: jpeg, pdf, tiff, eps, or mp. Your illustration will not be redrawn. Resolution of your illustrations must be at least 300 dpi, or, preferably, done as vector illustrations. If a text is needed in illustrations, use a font from the Times New Roman family in 11 pt.

Start: with the title in BOLD 14 pt, followed on the next line by the author(s)' name(s) in italic 11 pt.

Main Text: Use a font from the Times New Roman family in 11 pt.

End: with your name-address-email and your web site (if any).

Include: your high resolution small photo and a concise professional summary of your works and titles.

Please, submit your manuscripts to me at `asoifer@uccs.edu`.

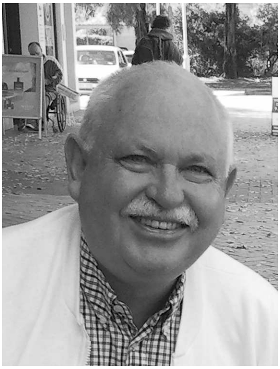
The success of the World Federation of National Mathematics Competitions (WFNMC), including its journal *Mathematics Competitions*, is in your hands — and in your minds!

Best wishes,

Alexander Soifer, Editor,
Mathematics Competitions
Immediate Past President, WFNMC

Examples of Mathematics and Competitions influencing each other

Peter James Taylor



Peter Taylor is an Emeritus Professor at the University of Canberra. He was a founder of the Australian Mathematics Trust and its Executive Director from 1994 to 2012. He holds a PhD at the University of Adelaide.

This paper gives examples of where developments in mathematics have enabled new types of competition problems or where problem solving activity with competitions has led to new results.

1. Australian Examples

I will start by citing examples of mathematics discovery via a problems committee in Australia, namely the committee for the Mathematics Challenge for Young Australians. Started in 1990 this is a rather unique competition in which school students are given 3 weeks to respond to up to six challenging questions. These questions are worth 4 points each, and have up to 4 parts worth 1 (or more) points. The problems committee members propose problems which are narrowed to a shortlist. In the deliberative stages committee members will work in small groups, normally 2 to 4 people, who explore the problem and develop it into final form. Often major changes, sometimes rendering the final wording barely recognisable from the original, take place. The small groups are also required to develop optional supplementary extension problems which may be quite difficult.

In the course of the years since this competition started committee members have discovered many new mathematical results and connected with advanced mathematical results. I discuss here four examples. Two led to refereed papers, one made (inadvertently) an IMO short list. The other was an innovative way of using a seemingly important abstract result and unpacking it to a problem which could be attempted by competent high school students.

1.1. O'Halloran Numbers

I start with a question which was called „Boxes“ and which was posed in 1996. The simpler version, for junior students, ultimately appeared as:

A rectangular prism (box) has dimensions x cm, y cm and z cm, where x, y and z are positive integers. The surface area of the prism is A cm^2.

1. Show that A is an even positive integer.
2. Find the dimensions of all boxes for which $A = 22$.
3. a) Show that A cannot be 8.
   b) What are the next three even integers which A cannot be?

and its solution as published was

1. For each face, the area in square centimetres is a whole number. Opposite faces have the same area. So the sum of the areas of the opposite faces is an even number. Then the surface area in cm^2 is even since the sum of 3 even numbers is even.

2. Here is a complete table guaranteeing all values of A up to 54 (as required in the Intermediate version of this question).

x	1	1	1	1	1	1	1	1	1	1	1	1	1
y	1	1	1	1	1	1	1	1	1	1	1	1	1
z	1	2	3	4	5	6	7	8	9	10	11	12	13
A	6	10	14	18	22	26	30	34	38	42	46	50	54

1	1	1	1	1	1	1
2	2	2	2	2	2	2
2	3	4	5	6	7	8
16	22	28	34	40	46	52

x	1	1	1	1
y	3	3	3	3
z	3	4	5	6
A	30	38	46	54

1	1
4	4
4	5
48	58

2	2	2	2	2
2	2	2	2	2
2	3	4	5	6
24	32	40	48	56

2	2	2
3	3	3
3	4	5
42	52	62

2
4
4
64

3
3
3
54

The table gives (1, 1, 5) and (1, 2, 3).

3. a) 8 does not appear in the table, so A cannot be 8.
   b) 12, 20 and 36 are the next even areas not listed.

The committee, particularly Andy Edwards and Mike Newman took a great interest in this and explored what other numbers could not be listed. After massive searching they found the 16 numbers 8, 12, 20, 36, 44, 60, 84, 116, 140, 156, 204, 260, 380, 420, 660, 924.

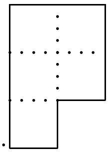
Further searching, as the number of possible values of A becomes denser, has so far found no more.

Peter O'Halloran, two weeks before he died, gave permission to name these numbers after him. Mike Newman took the matter up with a French number theorist. In their paper Louboutin and Newman **[6]** they pose an equivalent formulation, looking for solutions of the Diophantine equation $xy + yz + zx = d$, and unravelled the number theory, but whereas these numbers have an entry in the Online Encyclopaedia of Integer Sequences, no greater O'Halloran number is known after searching cuboids with areas up to 60 million or so.

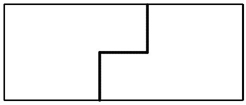
1.2. P Tiles

In 1998 Mike Newman and I were working on a problem which eventually appeared as

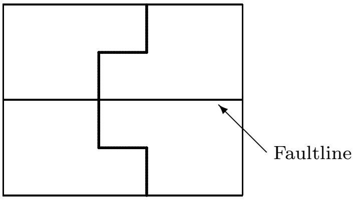
A P-tile is made from 5 unit squares joined edge to edge as shown. It can be used to tile some rectangles made up of unit squares.



For example, a 5 by 2 rectangle can be tiled by two P-tiles.



This tiling is said to be *fault-free* because there is no straight line going from one side of the rectangle to the other (beside its edges). However the following tiling of a 5 by 4 rectangle has a *faultline* and is not fault-free.

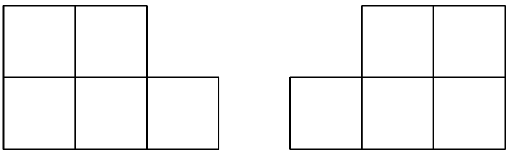


Note that the tiles can be placed with either face up.

1. Draw a fault-free tiling of a 5 by 4 rectangle using P-tiles.
2. Draw a fault-free tiling of a 5 by 6 rectangle using P-tiles.
3. Show that a fault-free tiling is possible for any 4 by m rectangle using P-tiles where m is a multiple of 5.
4. Show that a 5 by 3 rectangle cannot be tiled by P-tiles.

While working on this Mike and I discovered a further result which resulted in an extension problem.

If a $5 \times n$ rectangle can be tiled using n pieces like those shown in the diagram, prove that n is even.



Andy Liu devised the following neat solution.

Colour in red the first, third and fifth row of a tiled rectangle and colour in white the second and fourth row. We get $3n$ red squares and $2n$ white squares. Each copy of the figure can cover at most 3 red squares. It follows that each copy must cover exactly 3 red and 2 white squares. The shape of the figure implies that the two white squares are on the same row. Therefore a white line must have an even number of squares, i. e. n is even.

Note: Because the source of the problem had been inadvertently incorrectly annotated, this problem became proposed exactly as Problem C2(a) in the 1999 IMO Shortlist. When recognised at the Jury it was immediately disqualified as a known problem.

1.3. Spouse Avoidance

For the 2012 Challenge paper, Kevin McAvaney, Steve Thornton and I worked on the problem which eventually appeared as follows.

The Bunalong Tennis Club is running a mixed doubles tournament for families from the district. Families enter one female and one male in the tournament. When the tournament is arranged, the players discover the twist: they never partner or play against their own family member.

The tournament, called a TWT, is arranged so that:

(a) each player plays against every member of the same gender exactly once

(b) each player plays against every member of the opposite gender, except for his or her family member, exactly once

(c) each player partners every member of the opposite gender, except for his or her family member, exactly once.

Using the notation M_1 and F_1 for the male and the female in family 1, M_2 and F_2 for family 2 and so on, an example of an allowable match is M_3F_1 v M_6F_4.

1. Explain why there cannot be fewer than four families in a TWT.
2. Give an example of a TWT for four families in which one of the matches is M_1F_4 v M_2F_3.
3. Give an example of a TWT for five families in which one of the matches is M_1F_3 v M_2F_5.
4. Find all TWTs for four families.

Problems of this type have appeared over the years, and have been called Spouse Avoidance Mixed Doubles Round Robins (SAMDRR) after Anderson [**1**]. The solution as published included the following.

1. Each match involves two males and two females. Since the two members of a family never play in the same match, each match requires four players from four different families.
2. M_1F_4 v M_2F_3 $\quad$ M_2F_4 v M_3F_1 $\quad$ M_3F_2 v M_4F_1
   M_1F_2 v M_3F_4 $\quad$ M_2F_1 v M_4F_3
   M_1F_3 v M_4F_2
3. Using a tableau approach as in Part **2.**, systematic counting yields:

   M_1F_3 v M_2F_5 $\quad$ M_2F_4 v M_3F_5 $\quad$ M_3F_1 v M_4F_5 $\quad$ M_4F_3 v M_5F_2
   M_1F_2 v M_3F_4 $\quad$ M_2F_3 v M_4F_1 $\quad$ M_3F_2 v M_5F_1
   M_1F_5 v M_4F_2 $\quad$ M_2F_1 v M_5F_4
   M_1F_4 v M_5F_3
4. M_1 must partner F_2, F_3, F_4 and play against M_2, M_3, M_4. This gives the following two possibilities to build on:

(a) M_1F_2 v M_3F_4
M_1F_3 v M_4F_2
M_1F_4 v M_2

(b) M_1F_2 v M_4F_3
M_1F_3 v M_2
M_1F_4 v M_3

In addition, M_1 must play against F_2, F_3, F_4. This gives:

(a) M_1F_2 v M_3F_4
M_1F_3 v M_4F_2
M_1F_4 v M_2F_3

(b) M_1F_2 v M_4F_3
M_1F_3 v M_2F_4
M_1F_4 v M_3F_2

Case (a): M_2 must partner F_1 and F_4 and play against M_3, M_4, F_1 and F_3. This gives:

M_1F_2 v M_3F_4
M_1F_3 v M_4F_2
M_1F_4 v M_2F_3

M_2F_1 v M_4F_3
M_2F_4 v M_3F_1

Then M_3 must partner F_2 and play against M_4. This gives:

M_1F_2 v M_3F_4
M_1F_3 v M_4F_2
M_1F_4 v M_2F_3

M_2F_1 v M_4F_3
M_2F_4 v M_3F_1

M_3F_2 v M_4F_1

Case (b): M_2 must partner F_1 and F_3 and play against M_3, M_4, F_1 and F_4. This gives:

M_1F_2 v M_4F_3
M_1F_3 v M_2F_4
M_1F_4 v M_3F_2

M_2F_1 v M_3F_4
M_2F_3 v M_4F_1

Then M_3 must partner F_1 and play against M_4. This gives:

M_1F_2 v M_4F_3
M_1F_3 v M_2F_4
M_1F_4 v M_3F_2

M_2F_1 v M_3F_4
M_2F_3 v M_4F_1

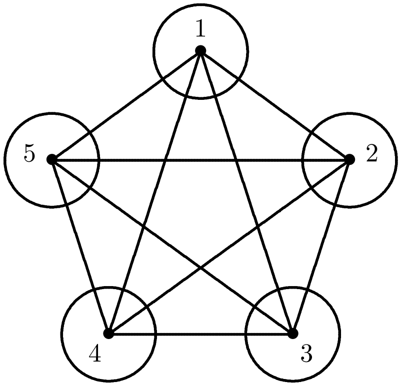
M_3F_1 v M_4F_2

It is easy to check that both of the completed schedules above satisfy all the conditions. Thus these two schedules are the only TWTs for four families.

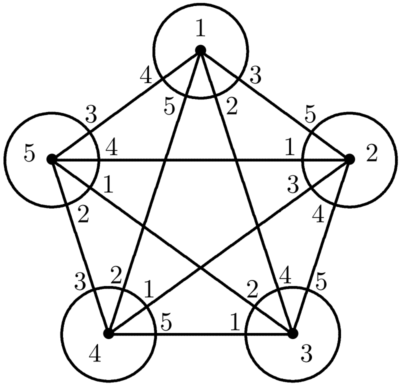
Further development

It has always been our intention, to help teachers assess, to provide alternative solutions. The group of us devised other methods, including a graph theory method. We devised a graphical method of representing the situations and solving the problem. For example an alternative method we devised for the solution of **3**. was as follows:

We use a graph. Each of the five males must play against all the other males so we draw five vertices labelled 1, 2, 3, 4, 5 to represent the males and join every pair of vertices with an edge to represent their matches.



In each match, each male must partner and play against a female that is not in his family. So on each end of each edge we place one of the labels 1, 2, 3, 4, 5 representing the two females in the match. The female label that is closest to a male vertex is that male's partner in the match. Again after some trial and error, we might get:



The bottom edge, for example, represents the match M_3F_1 v M_4F_5.

Further outcomes

Of the three of us Kevin's research field includes Graph Theory and this graphical representation inspired him to note an alternative partial solution to a famous problem of Leonhard Euler. In 1792 Euler proposed the following fairly simple 36 officers problem.

Given 6 officer ranks and 6 regiments, is it possible to arrange 36 officers in a square of 6 rows and 6 columns so that each row and each column contains exactly one officer of each rank and exactly one officer of each regiment.

This gets us into Latin Square territory. Euler is in effect asking if there is a pair of self-orthogonal Latin Squares. Whereas in the 20th Century it was resolved there are no pairs of orthogonal matrices of order 6 the proof is not easy. In 1973 a simpler algebraic and in 2011 a simpler graph theoretical method were found to show there are no self-orthogonal Latin Squares of order 6. The subsequent paper by McAvaney, Taylor and Thornton [**7**] provides an alternative graph theoretical verification of this result which is shorter than the 2011 result. The 1973 and 2011 results are referenced in [**7**].

1.4. Erdős Discrepancy Problem

In 1932 Paul Erdős posed a problem which became known as the Erdős Discrepancy Problem. It seems that he came across the idea while investigating Riemann's Hypothesis and zeta functions. There are various equivalent formulations. In 1957, Erdős [**3**] listed this among unsolved problems as important to him. The problem became frequently referenced in online blogs, particularly those of Gowers and Tao. It was eventually solved by Tao in 2015. One of the Gowers formulations is:

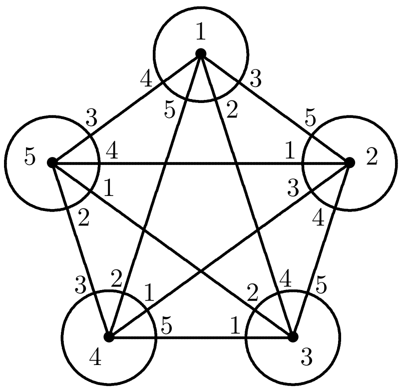
Statement of Problem: Is it possible to find a ± 1-sequence $x_1, x_2, \ldots$ and a constant C such that $|\sum_{i=1}^{n} x_{id}| \leqslant C$ for every n and every d?

This is an abstract formulation, involving discrepancies in arithmetic progressions. Steve Thornton, discussed above for his role in the Spouse Avoidance problem, started reading the blogs by Gowers and Tao and decided to try to bring this into the real world as one which could even be understood by (talented) high school students. He tried a few leads but eventually found a nice idea which resulted in the following, after work by several colleagues and fine-tuned by the Maths Challenge's Problems Committee Chairman Kevin McAvaney.

1.5. Statement of Problem in Mathematics Challenge

Two empty buckets are placed on a balance beam, one at each end. Balls of the same weight are placed in the buckets one at a time. If the number of balls is the same in each bucket, the beam remains horizontal. If there is a difference of only one ball between the buckets, the beam moves a little, but the buckets and balls remain in place. However if the difference between the number of balls is two or more, the beam tips all the way, the buckets fall off, and all the balls fall out.

There are several bowls, each containing some of the balls and each labelled L or R. If a ball is taken from a bowl labelled L, the ball is placed in the left bucket on the beam. If a ball is taken from a bowl labelled R, the ball is placed in the right bucket.



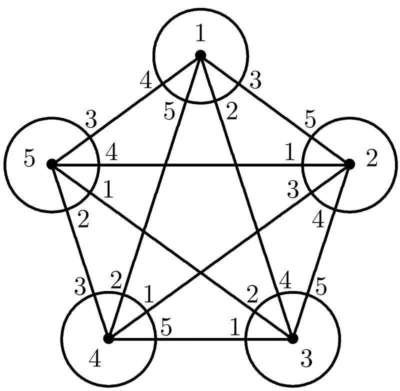
1. Julie arranges six labelled bowls in a row. She takes a ball from each bowl in turn from left to right, and places it in the appropriate bucket. List all sequences of six bowls which do *not* result in the beam tipping.
2. Julie starts again with both buckets empty and with six bowls in a row. As before, she takes a ball from each bowl in turn, places it in the appropriate bucket, and the beam does not tip. She then empties both buckets and takes a ball from the 2nd, 4th, and 6th bowl in turn and places it in the appropriate bucket. Again the beam does not tip. Once more she empties both buckets but this time takes a ball from the 3rd and 6th bowl in turn and places it in the appropriate bucket. Yet again the beam does not tip. List all possible orders in which the six bowls could have been arranged.

For a large number of bowls, a ball could be taken from every bowl, or every second bowl, or every third bowl, and so on. If a ball is taken from bowl m, followed by bowl $2m$, then bowl $3m$, and so on (every mth bowl), we say an *m-selection* was used. For example, in **2.**, Julie used a 1-selection, then a 2-selection, and finally a 3-selection.

3. Find all sequences of 11 bowls for which the beam does not tip no matter what m-selection is used.
4. Show that it is impossible to have a sequence of 12 bowls so that every m-selection is non-tipping.

Each part of the problem had multiple solutions listed. I provide the first alternative for each.

1. Moving through the sequence of bowls from the first to the last, the beam will tip if and only if the difference in the number of Ls and Rs is at any stage greater than 1. The following tree diagrams show the possible sequences, from left to right, of 6 bowls that avoid the beam tipping.



So the only sequences of bowls for which the beam does not tip are:
LRLRLR, LRLRRL, LRRLLR, LRRLRL,
RLLRLR, RLLRRL, RLRLLR, RLRLRL.

2. Since Julie uses all bowls and does not tip the beam, the bowls must be in one of the eight sequences found in Part **1.**:
LRLRLR, LRLRRL, LRRLLR, LRRLRL,
RLLRLR, RLLRRL, RLRLLR, RLRLRL.

Julie uses bowls 2, 4, 6 without tipping the beam. This eliminates the sequences LRLRLR, LRLRRL, RLRLLR, RLRLRL. So she is left with the sequences LRRLLR, LRRLRL, RLLRLR, RLLRRL.

Julie uses bowls 3 and 6 without tipping the beam. This eliminates the sequences LRRLLR and RLLRRL.

So the only sequences that work for all three procedures are: LRRLRL and RLLRLR.

3. The beam will not tip for any m-selection with $m \geqslant 6$ since, in those cases, a ball is drawn from only one bowl. For each m-selection with $m \leqslant 5$, the first 2 bowls must be RL or LR. For $m = 1$, let the first 2 bowls be LR.

Then, for $m = 2$, bowl 4 must be L. Hence bowl 3 is R.

Bowl	1	2	3	4	5	6	7	8	9	10	11
Letter	L	R	R	L							

For $m = 3$, bowl 6 must be L. Hence bowl 5 is R.

Bowl	1	2	3	4	5	6	7	8	9	10	11
Letter	L	R	R	L	R	L					

For $m = 2$, bowl 8 must be R. Hence bowl 7 is L.

Bowl	1	2	3	4	5	6	7	8	9	10	11
Letter	L	R	R	L	R	L	L	R			

For $m = 5$, bowl 10 must be L. Hence bowl 9 is R.

Bowl	1	2	3	4	5	6	7	8	9	10	11
Letter	L	R	R	L	R	L	L	R	R	L	

Bowl 11 can be L or R. Thus we have two sequences starting with LR such that no m-selection causes the beam to tip.

Similarly, there are two sequences starting with RL such that no m-selection causes the beam to tip.

So there are four sequences of 11 bowls such that no m-selection causes the beam to tip:

LRRLRLLRRLL, LRRLRLLRRLR,
RLLRLRRLLRR, RLLRLRRLLRL.

4. Suppose we have a sequence of 12 bowls for which every m-selection is non-tipping. Then, as in Part **3.**, up to bowl 10 we have only two possible sequences:

Bowl	1	2	3	4	5	6	7	8	9	10	11	12
Letter	L	R	R	L	R	L	L	R	R	L		

Bowl	1	2	3	4	5	6	7	8	9	10	11	12
Letter	R	L	L	R	L	R	R	L	L	R		

For $m = 6$, bowl 12 must be R and L respectively. However, for $m = 3$, bowl 12 must be L and R respectively. So there is no sequence of 12 bowls for which every m-selection is non-tipping.

I conclude by opining that this is a very nice way of setting a problem, that is, finding an abstract result and contextualising it in the real world.

2. New and Rediscovered Methods of Proof

Over recent years there has been excellent innovation in methods of proof. I will give one example of a rediscovered method (at least rediscovered in the West) and another nice method which evolved from 20th century research and student journal discussion.

2.1. Barycentric Coordinates

In 1989 I was trying to solve a new Tournament of Towns problem of geometric nature, but no Euclidean method offered hope. My Bulgarian colleague Jordan Tabov offered me a proof using Barycentric coordinates. Barycentric Coordinates are introduced in Coxeter [**2**] and Tabov and Taylor [**7**] and so I will assume the reader can access what they are elsewhere. They were introduced by Möbius in 1827 but they had been lost to Western syllabi and because they were rarely a prime method of solution in the IMO were not in training programs either.

But they have now somewhat been revived. They are particularly useful in collinearity questions. A difficult problem on the 1997 shortlist was of a collinearity nature and 3 dimensional. There were Euclidean proofs which justified a placement in the 1997 IMO as question 6 (i. e. very difficult). After a Bulgarian present at the IMO produced a short Barycentric proof the problem was dropped, and to my knowledge also

I don't think three dimensional geometry has been used since either, although that is another matter. I later became aware of a British IMO team member who often used Barycentric Coordinates in his proofs, much to the chagrin of his mentors I suspect.

The following problem appeared in the 1995 Bulgarian Mathematical Olympiad and is typical of those conducive to Barycentric treatment.

Let A_1, B_1 and C_1 be points on sides BC, CA and AB, respectively, of triangle ABC. If AA_1, BB_1 and CC_1 pass through the centroid G of triangle $A_1B_1C_1$, prove that G is the centroid of triangle ABC as well.

A Barycentric solution is as follows.

Let (g_a, g_b, g_c) be the barycentric coordinates of G with respect to triangle ABC. Then the barycentric coordinates of A_1 are

$$\left(0, \frac{g_b}{g_b+g_c}, \frac{g_c}{g_b+g_c}\right) = \left(0, \frac{g_b}{1-g_a}, \frac{g_c}{1-g_a}\right).$$

Similarly, the coordinates of B_1 and C_1 are

$$\left(\frac{g_a}{1-g_b}, 0, \frac{g_c}{1-g_b}\right) \quad \text{and} \quad \left(\frac{g_a}{1-g_c}, \frac{g_b}{1-g_c}, 0\right).$$

In this notation, if G is the centroid of $\triangle A_1B_1C_1$, then

$$\begin{aligned} g_a &= \frac{1}{3}\left(\frac{g_a}{1-g_b} + \frac{g_a}{1-g_c}\right), \\ g_b &= \frac{1}{3}\left(\frac{g_b}{1-g_a} + \frac{g_b}{1-g_c}\right) \\ \text{and} \quad g_c &= \frac{1}{3}\left(\frac{g_c}{1-g_a} + \frac{g_c}{1-g_b}\right). \end{aligned}$$

These equations are equivalent to

$$\begin{aligned} 1-2(g_b+g_c)+3g_bg_c &= 0, \\ 1-2(g_c+g_a)+3g_cg_a &= 0, \\ \text{and} \quad 1-2(g_a+g_b)+3g_ag_b &= 0. \end{aligned}$$

Adding these equations and taking into account that $g_a+g_b+g_c=1$, we get

$$3(g_bg_c+g_cg_a+g_ag_b)-1=0,$$

which can be rewritten in the form

$$3(g_bg_c+g_cg_a+g_ag_b)-(g_a+g_b+g_c)^2=0,$$

and

$$\frac{1}{2}(g_b-g_c)^2+\frac{1}{2}(g_c-g_a)^2+\frac{1}{2}(g_a-g_b)^2=0.$$

Consequently $g_b - g_c = g_c - g_a = g_a - g_b = 0$, and hence (in view of the fact that $g_a + g_b + g_c = 1$),

$$g_a = g_b = g_c = \frac{1}{3},$$

i.e. G is the centroid of $\triangle ABC$.

2.2. Method of the Moving Parallel

There were in the 20th Century a number of articles in various journals investigating dissections of regular polygons. Galvin [**4**] and Kasimatis [**5**] are merely two. Competitions, in particular the International Mathematics Tournament of Towns, have featured problems along the themes discussed. The solutions can involve moving parallel lines through the polygon.

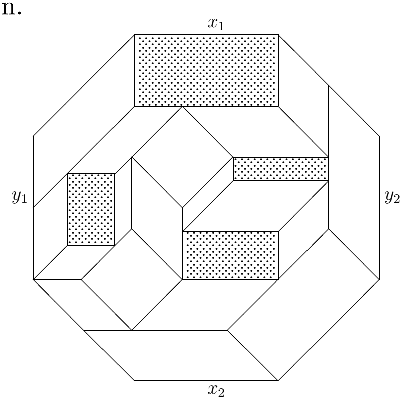
The following problem, from 1983 and which can be found in Taylor [**9, p. 62**], is an example.

(a) A regular $4k$-gon is cut into parallelograms. Prove that among these there are at least k rectangles.

(b) Find the total area of the rectangles in (a) if the lengths of the sides of the $4k$-gon equal a.

The following solution by Andy Liu illustrates the method.

(a) Let the regular $4k$-gon be dissected into parallelograms. Let x_1 and x_2 be a pair of opposite sides. The set of all parallelograms with one side parallel to x_1, starts from x_1 and eventually reaches x_2, possibly subdividing into several streams. The diagram illustrates the case of a regular octagon.



Since the regular polygon has $4k$ sides, there is a pair of opposite sides y_1 and y_2 perpendicular to x_1 and x_2. The set of parallelograms with one side parallel to y_1 starts from y_1 and eventually reaches y_2, again possibly subdividing into several streams. Now these two sets of parallelograms must cross each other. This is only possible at parallelograms with one pair of opposite sides parallel to x_1 and the other to y_1.

Since x_1 and y_1 are perpendicular, this parallelogram is actually a rectangle (due to subdividing into several streams, four such rectangles based on x_1, x_2, y_1, y_2 in the diagram exist and are shaded).

In the regular $4k-$gon, there are k sets of mutually perpendicular pairs of opposite sides. Hence there must be at least k rectangles in the dissection.

Note that in the diagram we can also identify a rectangle (in fact three exist) based on the two other pairs of opposite sides.

(b) Since the sides of the $4k-$gon are all of length a, the width of each set of parallelograms in (a), in the direction of the side of the $4k-$gon defining the set, is equal to a. It follows that the sum of the areas of all rectangles in the set is a^2. It follows that the total area of the rectangles is ka^2.

Note: Another very nice problem of this type, based around a regular hexagon, was set in the International Mathematics Tournament of Towns in 1989 and can be found in Taylor [**10, p. 12**].

Conclusions

These examples show a bridge between mathematical development and competitions. But the bridge also applies to the mathematicians themselves. Former winners of competitions have a high profile among leading researchers. Ten (at least to my knowledge) Fields Medallists (one declined) were IMO Gold Medallists.

Vladimir Drinfeld, USSR/Ukraine, IMO Gold Medal 1969, Fields Medal 1990.

Jean Christophe Yoccoz, France, IMO Gold Medal 1974, Fields Medal 1994.

Richard Borcherds, UK, IMO Gold Medal 1978, Fields Medal 1998.

(Sir) Timothy Gowers, UK, IMO Gold Medal 1981, Fields Medal 1998.

Grigori Perelman, USSR/Russia, IMO Gold Medal 1982, Fields Medal 2006 (declined).

Stanislav Smirnov, USSR/Russia, IMO Gold Medal 1986, 1987, Fields Medal 2010.

Terence Tao, Australia, IMO Gold Medal 1988, Fields Medal 2006.

Ng Bao Chu, Vietnam/France, IMO Gold Medal 1988, 1989, Fields Medal 2010.

Maryam Mirzakhani, Iran, IMO Gold Medal 1994, 1995, Fields Medal 2014.

Artur Avila, Brazil/France, IMO Gold Medal 1995, Fields Medal 2014.

Presumably this list becomes very dense if we look at correlations between competition winners and research citations.

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YES-WE-CAN! A History of Rolf Nevanlinna's name on IMU Prize

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Born and educated in Moscow, Alexander Soifer has for 40 years been a Professor at the University of Colorado, teaching math, and art and film history. During 2002-2004 and 2006-2007, he was Visiting Fellow at Princeton University. Soifer published ca. 400 articles and 13 books, including such books in Springer as The Scholar and the State: In the search of Van der Waerden; The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators; Mathematics as Problem Solving; How Does One Cut a Triangle?; Geometric Etudes in Combinatorial Mathematics; Ramsey Theory Yesterday, Today, and Tomorrow; The Colorado Mathematical Olympiad and Further Explorations: From the Mountains of Colorado to the Peaks of Mathematics; and The Colorado Mathematical Olympiad, The Third Decade and Further Explorations: From the Mountains of Colorado to the Peaks of Mathematics. He has founded and for 36 years ran the Colorado Mathematical Olympiad. Soifer has also served on the Soviet Union Math Olympiad (1970—1973) and USA Math Olympiad (1996—2005). He has been Secretary of the World Federation of National Mathematics Competitions (WFNMC) (1996—2008); Senior Vice President of the WFNMC (2008—2012); and President of WFNMC (2012—2018). He *is presently Editor of Mathematics Competitions, the journal you are holding in your hands. Soifer is a recipient of the Federation's Paul Erdős Award (2006); his Erdős number is 1.*

> It is my opinion that the tiniest moral matter is more important than all of science, and that one can only maintain the moral quality of the world by standing up to any immoral project.
>
> *Luitzen Egbertus Jan Brouwer*

During 1959—1962, the Finnish Mathematician Rolf Nevanlinna (1895—1980) was President of the International Mathematical Union (IMU), the highest organization in our profession. In 1981 IMU Executive Committee decided to create a Rolf Nevanlinna Prize for „Mathematical Aspects of Information Sciences", i. e., Mathematical Aspects of Computer Science. A year later Helsinki University, Finland, offered to pay for the prize (a gold medal with Nevanlinna's profile and cash to match the Field's Medal, or ca. $ 15,000 total). The IMU Executive Committee accepted the Finnish offer and has been giving the Rolf Nevanlinna Prize once every four years at the International Congress of Mathematicians (ICM), the last time on August 1, 2018 in Rio de Janeiro.

What was Nevanlinna's contribution to mathematical aspects of computer science, one may ask? The IMU Internet page answers:

> The prize was named in honor of Rolf Nevanlinna... who in the 1950s had taken the initiative to the computer organization at Finnish universities.

Is that all? How could IMU exhibit such an eclipse of common sense by taking „initiative to the computer organization at Finnish universities" for a major contribution to Theoretical Computer Science?

I had heard about this prize without paying any attention to the person of Professor Nevanlinna until my twenty-year long research for and writing of the book *The Scholar and the State: In Search of Van der Waerden* [1]. In studying the documents of the 1946 job search at Zürich University, I discovered that Rolf Nevanlinna and Bartel Leendert van der Warden were the finalists for the professorship. Nevanlinna got the job (while Van der Waerden had to wait until the next opening occurred in 1950).

I learned that Nevanlinna served as the *Rektor* of Helsinki University — till the end of the horrific World War II, when he was asked to vacate his position. Why, you may wonder? In his speeches and articles Nevanlinna praised Adolf Hitler as the liberator of Europe. Worse yet, Nevanlinna served as the Chair of the Finnish *S. S.* Recruitment Committee. You no doubt realize that *S. S.* was not a military force like *Wehrmacht*. *S. S.* was the military arm of the Nazi Party, responsible for most of the crimes against humanity, established at Nuremberg and other post-war trials. Nevanlinna's past was well known in Zürich, and one member of the Canton government challenged his University appointment. No problem, the University and government bent history to protect their reputations, together with the reputation of the dismissed in Finland University Rector Nevanlinna.

What has the IMU Executive Committee done? A common sense would tell these few distinguished officials from mathematics that they must pay attention to the moral bearings of the person, whose profile they etch on the medals. They certainly knew that Nevanlinna contributed nothing to theoretical computer science, and apparently were not bothered by that. Have the Executive members knowingly chosen a willing Nazi collaborator for the IMU Prize, or their ignorance is the protection of their integrity? Let us be charitable and presume ignorance of history until proven moral guilt.

I wrote all this and more about Nevanlinna in my 2015 book [1, pp. 189 and 286—288] and urged the IMU Executive Committee to change the name on the prize. But who reads 500-page books, and furthermore, who remembers a few pages after reading such a substantial dense volume?

Meanwhile, I was elected President (2012–2018) of the World Federation of National Mathematics Competitions (WFNMC) and as such was asked in July 2016 to give my organization's report to the General Assembly of the International Commission on Mathematical Instruction (ICMI) during its Hamburg quadrennial Congress. Right before my report, I had a brief exchange with the IMU President Shigefumi Mori:

– Mr. President, May I have your address, I would like to mail you a letter.

– What about?

– About one of your prizes.

– Which one?

– Rolf Nevanlinna Prize.

– You know, I cannot do anything by myself, but I will present your letter to the Executive Committee.

Today, looking back at this conversation, I have a feeling that President Mori knew what I was going to write about, for otherwise how would he know — without asking me — that I will complain about the name of the prize and profile on the medal?

I presented the report about the work of WFNMC over the preceding four years and then told the roomful of the delegates about Nazi collaboration of Rolf Nevanlinna. I ended with my personal impassioned call to change the name of the Rolf Nevanlinna Prize. A long silence fell on the room, followed by enthusiastic applauds.

In Hamburg-2016, as President, I called the meeting of the WFNMC Executive Committee. I told them about my intention to write a personal letter to IMU but would prefer to write on behalf of us all. I presented to them the relevant pages of my book [1] and asked for a vote. The Executives did not wish me to present this issue at the General Meeting of WFNMC (I still do not know why) but supported sending a letter to the IMU Executive Committee by a vote of 6 in favor, 1 abstained, and 1 against. My letter was approved by all 8 members of the Executive almost without changes, and off it went to IMU President Mori.

Mori acknowledged the receipt of the letter and promised to put it on the agenda of the next IMU Executive Committee meeting, 8 months later, in April-2017.

There were a few justifications for keeping the prize unchanged. To begin with, changing the prize's name would mean for IMU to acknowledge its mistake, which for many people and organizations is a hard act to perform. Money, donated by Finland, could have been another reason to keep the prize name unchanged. And so, I sent my second, this time a personal letter to Shigefumi Mori and the IMU Executive Committee, with two essential points. I offered to personally pay \$ 15,000 to IMU every four years to eliminate IMU's dependence upon Finnish funding. For someone, who started his American life from scratch as a refugee (legal, I shall add, to avoid temper tensions of President Trump), this was a serious expense, which, as a proverb observes, put my money where my mouth was. I also observed that while the 1981 IMU Executive Committee could have pleaded ignorance, now they could not do so, for I informed them of the Nazi collaboration of Rolf Nevanlinna. Keeping the Nevanlinna name on an IMU prize would stain Mathematics forever, I concluded.

Geombinatorics readers know [2] that IMU has been keeping dates and locations of its Executive Committee meetings a tightly guarded secret. I would have preferred *glasnost*, a Mikhail Gorbachev's word that translates as openness and transparency. Why does a discussion of a prize name change have to be conducted behind closed doors?

Only in late April-2017, did I learn that their meeting took place on April 1–2, 2017 in London, and asked President Mori to share their decision. His April 24, 2017, reply was a riddle. On the one hand, he wrote,

> We did discuss the issue regarding the Nevanlinna Prize at our recent EC meeting, and we made a decision.

On the other hand, he was not going to disclose that decision to me:

> But, as I am sure you understand, we need to discuss this with the partners involved. Before we have reached an agreement with them, we will not go public. We ask for your understanding of this way to proceed.

I met this part without understanding. „What if you do not reach an agreement with partners?“ I asked Mori, who went non-communicado for what felt like an eternity.

On August 10, 2018, IMU President Mori, reappeared:

> Dear Professor Soifer,
>
> This is to let you know of the decision that IMU has finally made at GA [General Assembly of IMU, July 30—31, 2018].
>
> It is the Resolution 7 of the attached „RESOL2018.pdf“, which you can also find under the item Resolutions of „18th GA in São Paulo, Brazil“ in the URL `https://www.mathunion.org/organization/general-assembly`
>
> Best regards,
> Shigefumi Mori
> President of the International Mathematical Union
> Phone: +81-75-753-7227

Let me reproduce for you Resolution 7 approved by the IMU's General Assembly:

Resolutions of the IMU General Assembly 2018
Resolution 7

The General Assembly requests the 2019–2022 IMU Executive Committee, giving due consideration to all the issues involved, to determine and set up statutes for a prize continuing and with the same purpose and scope as the Nevanlinna Prize but with a new name and appropriate funding to be secured. The statutes of the new prize will be sent to the Adhering Organizations for approval by a postal ballot.

It is ironic that the last Rolf Nevanlinna Prize was presented to the deserving winner the next day after the name change decision was made.

As you can see, one person and one organization, empowered by truth and glasnost, can affect a major change. This allows me to use Barack Obama's slogan in the title of this essay: Yes-We-Can!

I hope this time IMU will choose for this prize the name of a great theoretical computer scientist of high moral standards, such as John von Neumann, Norbert Wiener, or Claude Shannon.

It is surreal to see thousands of otherwise good people exhibit moral relativism and often hide behind tired slogans like „What can I do alone!“ Is silence truly golden? The most sensitive among us, such as Grigory Perelman, walk away from the Profession populated by the majority, for which Mathematics is all that matters, and ethics of the Profession matters not. This majority pumpers itself by calling the departed „crazy“ without realizing that crazy is the majority.

The leading mathematician of the United States believed that while in my call for the name change, I was „making some good points, the chances of IMU changing anything are very slim“. I thought so too. However, we ought to do all that we can to raise our Profession to a higher moral ground. Anything less than that would compromise our integrity and guarantee the victory of the status quo in this world that needs so much change. L. E. J. Brouwer rightly believed that „*the tiniest moral matter is more important than all of science*“. In my opinion, Ethics is not a minor external fare but an essential inner part of our Profession.

I am grateful to the July 29-30, 2018, General Assembly (GA) of the International Mathematics Union (IMU) for following our urging and deciding to replace the name of Rolf Nevanlinna on its prize and medal. However, GA erred in granting criminal tyrant Putin a propaganda tool by selecting Russia for hosting ICM-2022. War on Georgia, annexation of Crimea, War on Ukraine, etc., etc., etc. are the gravest violations of the international law, and must not be rewarded.

I do not think that for countries to order their mathematicians not to attend is right. I support the boycott on the individual level, by those who are not only mathematicians, but also human beings of high moral standards.

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Theoretical or foundational considerations for mathematics education as related to competitions

María Falk de Losada

María Falk de Losada has worked in mathematical competitions for almost forty years, cofounding the Colombian Mathematics Olympiad and the Iberoamerican Mathematics Olympiad both for secondary students (1981, 1985) and for university students (1997, 1998). She is a founding member of WFNMC, recipient of its David Hilbert Prize, and has been regional representative, vice president, president, chair of the awards committee, and continues to collaborate with the Federation as a member of its executive committee. She currently heads the masters and doctoral programs in mathematics education of the Universidad Antonio Nariño in Bogotá.

Mathematicians had their foundational dreams shattered in the twentieth century. Nevertheless, alternative strategies, including an axiomatic approach on the local level and structuralism in the larger global picture, were devised to give coherence to mathematics as a body of knowledge.

The fact that mathematics education is striving to establish itself and find its identity as an accepted scientific discipline can be seen clearly in the many topic study groups at ICME 13 that showed clearly that they were attempting to root practice in theory. A large number of TSGs stated as one of their aims the development of theoretical and conceptual frameworks, or of theoretical and methodological tools. The task design group spoke of „task design principles and theoretical approaches“, while the resource group hoped „to bring to the foreground and examine various theoretical and methodological approaches used to study resources“. Another group refered to the delimitation of a theory for their area „drawn from theoretical and epistemological perspectives“, while still another aimed to „specify the theoretical frame or rationale by which the selection of methodology and methods can be justified“.

Several Topic Study Groups went further, speaking of foundations. For example, TSG 19, „Problem solving in mathematics education“ underlined the necessity for establishing foundations for their area, which they define as answering the following questions. „What is required to support a research program in mathematical problem solving? What principles are important to relate problem solving activities and learners' construction or development of mathematical knowledge?“ And TSG 51 „Diversity of theories in mathematics education“ stated that „our group continues the efforts of the mathematics education community to consolidate and compactify the theoretical foundations of the domain“.

In our brief and somewhat tentative considerations in this article, we would like to argue that the area of mathematics education devoted to mathematics competitions can, in fact, cite a long and rich history that has built, if not foundations, then certainly theoretical and methodological frameworks. Our purpose in so doing is to begin a conversation leading to an analysis of these frameworks and their importance for situating and informing research in the field of competitions.

In preparing to do so, we have initiated an analysis of the theoretical frameworks or foundations for that part of mathematics education that is involved in competitions and we would like to share our thoughts. We maintain that mathematics competitions have engendered thinking that encompasses their methodological, epistemological and mathematical foundations.

Let us begin to explore this claim by asking: What was the relationship of mathematics to mathematics education at the time mathematical problem solving competitions reappeared in the second half of the nineteenth century in the setting of school mathematics? We believe that the most relevant feature was the increasing specialization in mathematical research, the necessity of studying a specialized branch of mathematics deeply in order to do research and solve original mathematical problems. In other words, research in mathematics itself had gone beyond the scope of what both *aficionados* from the general public and students could understand and work on. We believe this is what led both mathematicians and mathematics teachers to look for interesting and challenging problems that young minds could understand and solve, and become involved, sometimes passionately involved, in mathematics and in solving mathematical problems. It seems to us that it is not a coincidence that, at the same time that mathematical problem solving competitions began to be organized in schools, from Lewis Carroll to Martin Gardner there was also a revival and renewal of recreational mathematics for the general public, appearing in newspapers, magazines and popular books.

Mathematical problem solving competitions, as a branch of mathematics education, have a feature that distinguishes the work being done from every other initiative in the field. And this has its roots in Hungary at the Eőtvös and Kürschák competitions and the journal of problems in mathematics and physics, *Középiskolai Matematikai Lapok* or KöMaL. With common roots in these pioneering competitions, a school was formed that produced outstanding figures in mathematics, in methodology and in epistemology. Beginning with the work and leadership of Lipót Fejér (Leopold Weiss) who grew up solving problems from KöMaL and who placed second in the Eötvös competition of 1897, a school was formed that came to include, in varying degrees, Paul Erdős, George Pólya and Imre Lakatos, the great mathematician and collaborator with mathematicians around the globe, the influential thinker on problem solving and method, and the philosopher–epistemologist who dared to question formalist mathematics proposing an alternative interpretation of the character, origins, structure and justification of mathematical knowledge and its historic evolution. These three stand out among the many great Hungarian mathematicians whose mathematical formation began in or was intimately related to the competitions, especially because they migrated to England and the United States and worked and published in English, thus opening their ideas and results and bringing them to bear on the worldwide community of mathematicians and mathematics educators. In what follows we outline their contributions.

Lipót Fejér (1880–1959), precursor

Once competitions devoted to solving challenging mathematical problems were well established in Hungary, a new school of mathematics began to take form in that country rooted in the competitions. One of the first winners of the Eötvös Competition was Leopold Weiss (Lipót Fejér), and he was destined to become one of the mathematicians who were highly instrumental in forming the new generations of the Hungarian school.

Fejér's attitude towards mathematics changed dramatically in secondary school when he began solving problems from KöMaL and in 1897, the year in which he graduated from secondary school in Pécs, Fejér won second prize in the Eötvös Competition. That same year, Fejér began his studies in the Polytechnic University of Budapest where he studied mathematics and physics until 1902. Among his professors in Budapest were József Kürschak and Lórand Eötvös, whose names are well known to all who have worked in mathematics olympiads.

Pólya said the following of Fejér [1].

> Why was it that Hungary produced so many mathematicians in our time? Many people have asked this question which, I think, cannot be fully answered. However, there were two factors whose influence on Hungarian mathematics is clear and undeniable, one of these was Leopold Féjer, his work, his personality. The other factor was the combination of a competitive examination in mathematics with a problem solving journal.

In what follows we offer some considerations to answer the question: What is the relationship between Erdős, Pólya and Lakatos and why is it important to the ideas we wish to express?

Paul Erdős (1913—1996), the great mathematician, problem poser and solver

Erdős got his first formation in mathematics from his parents who were mathematics teachers. He won a national competition in problem solving (Jószef Pelikán has informed us that it was not the Eőtvős), which allowed him to study mathematics at university. He wrote his thesis under the direction of Fejér. He was awarded a postgraduate scholarship in Manchester and then in Princeton.

Asked if he believed that his mathematical development had been influenced by *Középiskolai Matematikai Lapok (KöMaL)?* Erdős answered (1993) [2]: „*Yes, of course, you really learn to solve problems in KöMaL. And many good mathematicians realize early on that they have mathematical ability*“.

Asked to what he attributed the great advance in Hungarian mathematics, Erdős said (1985) [1]: „*There must have been many reasons.*

There was a journal for secondary school, and the competitions, that began before Féjer. And once they began, they self perpetuated up to a certain point... But such things probably have more than a single explanation...“

Erdős’ contributions were great in quantity and importance, and cover a great range of topics. Erdős was primarily a problem solver, not a theory builder. He was attracted principally by problems in combinatorics, graph theory and number theory. For Erdős a proof must provide insight into why the result is valid and not be only a sequence of steps that lead to a formal proof without providing understanding.

Several of the results associated with Erdős had been proved previously by other mathematicians. One of these is the prime number theorem: *the number of primes not exceeding x is asymptotic to $x/\ln x$.*

The theorem had been conjectured in the eighteenth century, Chebyshev got close to a proof, and it was proved in 1896 independently by Hadamard and de la Vallée Poussin using complex analysis. In 1949 Erdős and Atle Selberg found an elementary proof, one which did indeed provide insight into why the theorem was true. The result is typical of the kind of mathematics that Erdős worked on. He proposed and solved problems that were elegant, simple to understand, but very difficult to prove.

In 1952 Erdős received the Cole Award from the American Mathematical Society for his several results in number theory, but in particular for his article *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem* which contained this very proof.

George Pólya (1887–1985), mathematician, educator and methodologist

At university in Budapest, Pólya learned physics with Lórand Eötvös and mathematics with Fejér. Pólya stated [1]:

> Féjer was a great influence on me, as he was on all of the mathematicians of my generation, and, in fact, once or twice I collaborated with Féjer on small things.

Pólya was awarded his doctoral title in mathematics in the academic year 1911–1912 having studied, essentially without supervisión, a problem in geometric probability. Pólya explained why he approached mathematics in a way that differed from the usual treatment, for example, in the problem-solving book in analysis that he wrote jointly with Gabor Szegö, Pólya had the idea of grouping the problems according to the method of solution used rather than the usual grouping by topics, in the following terms [1]:

> I came to mathematics very late... as I got close to mathematics and began to learn something about it, I thought: Well, this is true, I see, the proof seems conclusive, but how is it that people can find such results? My difficulty in understanding mathematics was how was it discovered?

Although the book of solutions to problems of analysis that he wrote with Szegö was a masterpiece that would make both authors famous, Pólya continued to look for answers to this question, publishing his three well-known works: *How to solve it*, *Mathematics and plausible reasoning* (1954), and *Mathematical discovery* in two volumes (1962, 1965).

Pólya maintained that in order to work on problem solving it is necessary to study heuristics, stating [6] that

> the purpose of heuristics is the study of the rules of discovery and invention... As an adjective heuristic means 'that which allows one to discover'... its purpose is that of discovering the solution to a problem that one is studying... What is good education? It is systematically giving the student the opportunity to discover for himself.

Speaking in general about teaching, Pólya said [9]:

> Teaching is not a science, it's an art. If it were a science there would be a best way of teaching and everyone would have to teach that way. Since teaching is not a science, there is room to accommodate different personalities... Let me say what I think teaching is. Perhaps the first point, which is widely accepted, is that teaching should be active, or better that learning should be active... The main point of teaching mathematics is developing problem solving strategies".

Imre Lakatos (1922–1974), philosopher and epistemologist

In 1953 Lakatos was supporting himself translating mathematical books into Hungarian. One of the books he translated at that time was Pólya's *How to Solve It*.

In the Hungarian Revolution of 1956 and the Soviet repression that followed, Lakatos realized that he was about to be imprisoned and escaped to Vienna, going from there to England where he began a doctoral program in philosophy. The ideas of Popper and Pólya were greatly influential in his work and his thesis *Essays on the Logic of Mathematical Discovery* was completed in 1961. It was following a suggestion of Pólya that the thesis took its theme from the history of Euler's formula $V - E + F = 2$. Lakatos never published his thesis as a book because he intended to improve it. In 1976, after his death (1974) the book was published by J. Worrall and E. G. Zahar (eds.), *I. Lakatos. Proofs and Refutations: The Logic of Mathematical Discovery.*

Worrall [7] described this work in the following terms.

> The thesis of 'Proofs and Refutations' is that the development of mathematics does not consist (as conventional philosophy of mathematics tells us it does) in the steady accumulation of eternal truths. Mathematics develops, according to Lakatos, in a much more dramatic and exciting way — by a process of conjecture, followed by attempts to 'prove' the conjecture (i.e. to reduce it to other conjectures) followed by criticism via attempts to produce counter-examples both to the conjectured theorem and to the various steps in the proof.

Of high importance to the present analysis with regard to *Proofs and Refutations* is its attack on formalism in the style of Hilbert, although it is worthwhile noting that Hilbert himself always recognized the importance of singular, unique problems in attracting young minds to mathematics, and his famous list of problems was made known precisely with that objective in mind.

An article that Lakatos wrote and that was originally published in *The Mathematical Intelligencer* [5] entitled, *Cauchy and the Continuum: The Significance of Non-Standard Analysis for the History and Philosophy of Mathematics* shows, in Hersh's interpretation [4], the objective that Lakatos pursued in his approach to the history of mathematics.

> The point is not merely to rethink the reasoning of Cauchy, not merely to use the mathematical insight available from Robinson's non-standard analysis to re-evaluate our attitude towards the whole history of the calculus and the notion of the infinitesimal. The point is to lay bare the inner workings of mathematical growth and change as a historical process, as a process with its own laws and its own 'logic', one which is best understood in its rational reconstruction, of which the actual history is perhaps only a parody.

With these three figures, formed in one way or another in the Hungarian school of mathematical problem solving competitions, the key ingredients have been readied that constitute the theoretical framework or possible foundations of the mathematics education related to competitions and that would allow a change in the way that mathematics is being done or, at least, a change in the mathematics that is being done. First, a prolific mathematician, foremost a problem solver rather than a theory builder, who worked his entire life with mathematicians throughout the world. That is to say, a view of the nature of mathematics. Second, an epistemologist who theorized about the nature of mathematical knowledge and broke with the formalist tradition that had dominated mathematics for much of the twentieth century. That is to say, a view of the nature of mathematical knowledge. Third, a methodologist who led change on the level of education. That is to say, a view of how such mathematics can (and should) be learned.

Perhaps, given the theoretical and foundational issues that were raised by many of the Topic Study Groups at ICME13, it is important for WFNMC and its members to recognize these underpinnings of their work in producing the mathematical challenges of competitions.

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What kind of math reasoning may help students in multiple-choice competition

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Abstract

The paper presents a personal view point on what students are expected to do at a multiple-choice format competition. Most of the examples are taken from the competition papers of *Chernorizec Hrabar* Math Tournament in Bulgaria.

Keywords: multiple-choice competitions, Chernorizec Hrabar tournament, mathematical reasoning

Introduction

The classic view point on the process of problem solving is given by George Polya in [1]. A dialog of 5 parts provides guideline for attacking math problem by (more or less) experienced person. The time limit is not among the conditions of implementation the Polya's advices. Another feature of this method is the attention to the details at any step. Sometimes the problem-solver is suggested to make several iterations to one and the same point of the process, to rethink the initial idea and so forth. Author's personal experience in the implementation of Polya's method refers to the long term tutoring of advanced students. Concerning multiple choice math competitions (McMC) the problems book [2] illustrates in excellent manner how Polya's plan could work when studying advanced school math. One can find in this book most of the competition topics developed in problem series. There are alternative solutions, hints, open problems — in general, everything student needs to prepare himself for the Australian (and not only) math competition.

Now suppose a student has passed a good training in problem solving. Does it guarantee him success at McMC? In our opinion even having solid background in problem-solving one could stumble over the appalling combination of McMC parameters.

Specifics of McMC and action plan

Multiple-choice math competitions (McMC) have two main features. The students are expected to:

- get many test items for very limited time;
- point the correct answer without giving any reasoning.

For instance the competition papers for 9—12 grade students of the Chernorizec Hrabar tournament (ChH) are composed of 30 test items (TI) and the time allowed is 90 minutes. This gives 3 minutes per TI in average. Let us count what kind of activities are expected to be done for these 3 minutes:

(1) to grab the question that means: to read, to understand, to feel it and to connect the question data with the data from the answer variants;

(2) to perform adequate actions, e.g. calculations, sketching;

(3) to take a decision, i.e. to chose among the answer variants or leaving blank, estimating the risk.

We will call the above 3-step plan McMCP and we consider it as an alternative of the George Polya's 4-step plan for problem solving. In fact, McMCP reflects a kind of synthetic student's cognitive activeness, while the Polya's plan refers to an analytic (creative) process. In our opinion Polya's plan is not applicable in any of its steps during an McMC. However, a lot of Polya's recommendations are implicit parts of McMCP and we will point this further.

The next example illustrates what we have in mind. Further the coding of problems **YYYY-A,B** means 'year-and-classes-from-A-to-B'.

2018-9,10. $\sqrt{\dfrac{x-1}{x}} = \dfrac{1}{x} \iff$

(A) $x_{1,2} = 1 \pm \sqrt{2}$ (B) $x_{1,2} = \dfrac{1 \pm \sqrt{5}}{2}$ (C) $x_{1,2} = 1 \pm \sqrt{5}$

(D) $x_{1,2} = \dfrac{1 \pm \sqrt{2}}{2}$ (E) none of these

Reasoning. Every time '$-$' derives negative value in (A)–(D), but

$$\sqrt{\frac{x-1}{x}} \geqslant 0 \Rightarrow x > 0.$$

Answer. (E).

Solution. We determine the radical domain:

$$\frac{x-1}{x} \geqslant 0 \iff x \in (-\infty; 0) \cup [1; +\infty).$$

The range of the radical is subset of $[0; +\infty)$, hence $\dfrac{1}{x} > 0$. For $x \in [1; +\infty)$ we have

$$\sqrt{\frac{x-1}{x}} = \frac{1}{x} \Rightarrow \frac{x-1}{x} = \frac{1}{x^2} \Rightarrow x(x-1) = 1 \Rightarrow x = \frac{1+\sqrt{5}}{2}.$$

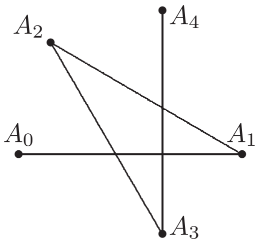
Comment. We will distinguish (the equation and its solution), which is a regular math task, from (the equation, answer variants and reasoning), which is a TI. It is clear that a trained student can find quickly the correct answer, let say for a minute. So he saves time for the harder TIs. On the contrary, a conscientious problem-solver may lose precious time in clarifying details in this routine solution.

About the McMC-style reasoning

In this section we will show by examples how different could be the reasoning during McMC from a canonical problem solving.

2017-9,10. We call 'move' the following action of the turtle in the plane: moving 10 units forward then turning 150° to the left. Initially the turtle is in point A_0; it makes 1 move and goes to point A_1; then it makes another move to point A_2, and so on. Find the angle $A_0A_{2017}A_{1275}$.

(A) 30°
(B) 45°
(C) 60°
(D) 90°
(E) none of these.



Reasoning. The intuition hints that the process seams to be cyclic: after several moves the turtle returns in the initial location. The picture shows that A_n lies on the circumcircle of triangle $A_0A_1A_2$. After a move A_nA_{n+1} rotates about the circumcentre on 150°. Since $\mathrm{LCM}(360; 150) = 1800$ and $1800 = 12 \cdot 150$ then $A_n \equiv A_{n+12}$. Hence $A_0A_{2017}A_{1275} \equiv A_0A_1A_3$. From the picture we get $\angle A_0A_1A_3 = 45°$.

Answer. (B).

Comment. This TI was the last one in the competition paper, that means it is hard. The reasoning is a bunch of an observant eye, sense of mathematics, and math knowledge. The McMCP is as follows.

(1) The students are expected to imagine that the turtle's trajectory consists of segments that are consecutive images of A_0A_1.

(2) The simple calculations reduce the desired angle to the one that could be seen at the picture.

(3) Finally, to recognize the correct answer one should discern the symmetry about the segment bisector of A_1A_3.

We challenge the reader to write a comprehensive solution of this problem (the author has never done it) and to compare with the McMC-style reasoning.

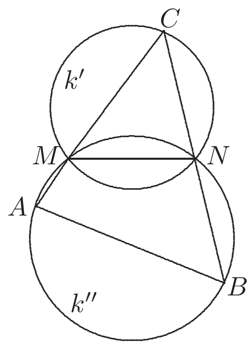
2017-9,12. The Sun raises just opposite my window in September, projecting the window into a bright rectangle on the wall. The wall is 4 meters apart the window. At what approximate distance in centimeters moves the rectangle for 10 minutes?
(A) more than 20 (B) between 20 and 15 (C) between 15 and 10
(D) between 10 and 5 (E) less than 5

Reasoning. The Sun circles round the Earth for 24 hours, hence it angular speed is $15°/h = (1/4)°/min$. The distance passed along a circle with radius 4 m at this angular speed for 1 minute equals $\dfrac{2\pi \cdot 400}{4 \cdot 360} \approx \dfrac{628}{360}$ cm. The arc of the circle corresponds to a small central angle, hence it length is close to the segment passed by the rectangle. Since $15 \cdot 360 < 6280 < 20 \cdot 360$, then (B).

Comment. The McMCP calls for a composition of common knowledge and estimation ability. Usually the problem-solving is related to a pure math model: a vehicle travels from A to B with constant speed etc. Here the real-life situation requires a math modeling that takes into account natural phenomenon and several assumptions, e.g. the Sun raises straight up for a 10 minute period in the morning; the movement on the wall is almost the same as the movement along a circular arc. The type of the answer variants allows accepting such approximations for the sharp parameters of the physical process. Thus, the TI checks the synthetic thinking of the students. Polya's recommendations to draw a picture, and then to rethink the details are very helpful part of the McMCP.

2018-11,12. In the figure $MN = 1$, $\widehat{MCN}$ equals $\frac{3}{4}$ of the circle k' and $\widehat{MAN}$ is $\frac{5}{6}$ of the circle k''. Find the length of AB.

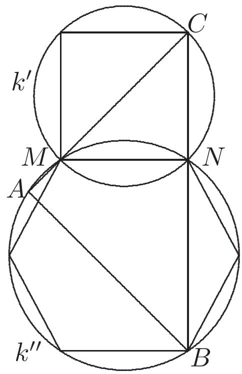
(A) $\dfrac{\sqrt{6}+\sqrt{2}}{3}$
(B) $\dfrac{\sqrt{6}+\sqrt{3}}{3}$
(C) $\dfrac{\sqrt{6}+\sqrt{3}}{2}$
(D) $\dfrac{\sqrt{6}+\sqrt{2}}{2}$
(E) none of these



Reasoning. The circles k' and k'' are circumscribed around a square and a regular hexagon, having common side MN. The formulation of the question hints at the length of AB does not depend on the location of C on k'. Let us choose C to be that vertex of the square for which $NC = 1$. Hence $NB = \sqrt{3}$ as diagonal of the hexagon. Since $\triangle MNC \sim \triangle BAC$ then

$$AB = \frac{NM \cdot BC}{MC} = \frac{1 \cdot (1+\sqrt{3})}{\sqrt{2}}.$$

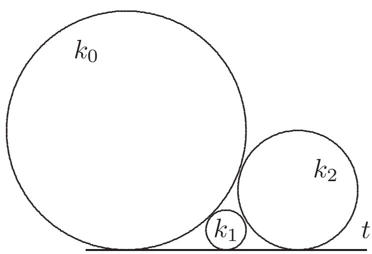
Answer. (D).



Comment. The proof that the length of AB does not depend on the location of C on k' is based on the equality $\angle ACB = \frac{1}{2}(\widehat{AB} - \widehat{MN}_{k''})$ and the fact that $\angle ACB = \angle MCN = \frac{1}{2}\widehat{MN}_{k'}$ does not depend on the location of C on k'. Further the Polya's advice to consider particular case is crucial: including the segments of the proportion $\frac{AB}{BC} = \frac{NM}{MC}$ into familiar polygons allows calculating their lengths.

2018-9,12. In the figure: circle k_0 touches line t; circle k_1 touches circle k_0 and line t; circle k_2 touches k_0, k_1 and t; circle $k_3 \neq k_1$ touches k_0, k_2 and t; circle $k_4 \neq k_2$ ... Given the radii of k_0 and k_1 equal 100 and 1 respectively find the miximum number of circles k_n that could be drawn in the same manner (including k_1 and k_2).

(A) below 5
(B) between 6 and 8
(C) between 9 and 12
(D) between 13 and 20
(E) between 21 and 100



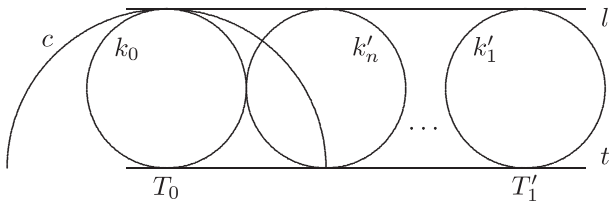
Reasoning. Denote by T_j the tangent point of k_j and t. Let c be the circle centered at T_0 with radius 200. Consider inversion I about c. We have $I(t) = t$, $I(k_0) = l$, $l \| t$. Circles $I(k_j) = k'_j$ are tangent to the lines l

and t so they are congruent. Since

$$T_0T_1 = \sqrt{(100+1)^2-(100-1)^2} = 20,$$

then for the point $I(T_1)=T_1'$ we find $T_0T_1'=\dfrac{200^2}{20}=2000$. The diameter of k_j' equals 200, thus between k_1' and k_0 are accommodated exactly 9 circles k_j' and with k_1' they are 10 in total.

Answer. (C).

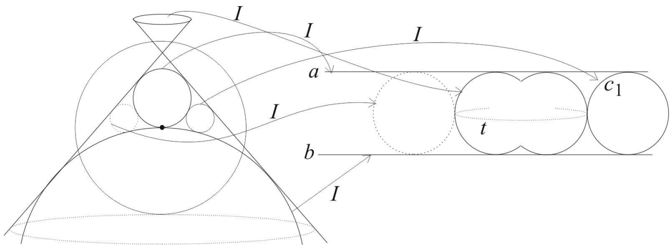


Comment. In fact the reasoning is an extraction from the solution. However, the McMC style here allows student to focus on the main idea skipping a lot of routine details. When the proper inversion is discovered, no obstacles stay on the way to the correct answer (see the Polya's advice about the lucky idea).

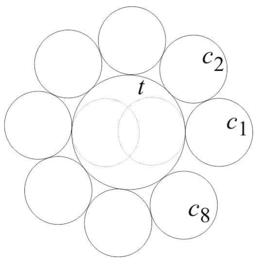
The next problem is not among the ones included in ChH competition papers. The author stated it during the VIII WFNMC Congress in the break between the Kiril Bankov's talk on inversion and McMC workshop session, so he feels his duty to explain why the problem worth some attention.

2018-WFNMC. Two spheres are placed in a cone, touching each other and the cone. How many at most are the spheres of maximal size that could be placed between the given spheres and the cone?

Reasoning. Let us clarify that the spheres of maximal size are touching the cone and the given spheres, i.e. they are inscribed in the room between the cone and the two spheres. Consider inversion I about any sphere centered at the touching point of the two given spheres. The images of these spheres after I are two parallel planes a and b; the image of the conic surface is a self-intersecting (spindle) torus t between a and b touching them. The images c_j of the spheres we study are spheres that touch externally t and the planes a and b. The left part of the figure presents a section of the initial construction by a plane through the axis of the cone (the bold point is the pole of the inversion); the right part shows what happen after the inversion.



Suppose the distance between a and b is 2. Then the radii of c_j are 1 and the diameter of t is between 2 and 4. The figure below shows the section of the inverted construction by a plane in the middle of a and b.



Now the question is 'How many at most are the circles with radii 1 that could be placed in a chain around a circle with radius r, $1 < r < 2$?' The answer could be obtained by a sketch using only a compass. The maximal number of inscribed spheres c_j equals 6 in the extremal case $r \to 1$ (when t is close to a sphere). The extremal case $r \to 2$ (when t become a horn (touching itself) torus) gives 9 for the maximal number of inscribed spheres.

Answer. Between 6 and 9 (both included).

Comment. This problem shows the advantages of the synthetic reasoning versus rigor math proof. It could be the jewel of a McMC paper as well as the horror of a math Olympiad. However, we did not include it in the ChH themes being apprehensive of inability of our best senior students to use compass and perform accurate figures during a competition :)

Concluding notes

The author's standing point is that the McMCs are rather a place for finding correct answers than for real math problem solving. Rigor math proofs are beyond the abilities of the largest part of the McMC participants. The participants at the final round of Bulgarian Math Olympiad (who are real problem-solvers) are just a small fraction of the participants at ChH or any other Bulgarian McMC. However, there are many among these students who have all the qualities of bright modern person, including math reasoning. The carrier in mathematics is attractive for a few of them, but mathematics as an instrument to polish the sharp mind is recognized of all our flock. The synthesis of math knowledge, math sense, sharp eye, evaluating the risk and taking proper decision is crucial for success in a McMC. It is also helpful for success in society.

About the TI in the paper

The 2017 ChH problems can be found in [3]; 2018-9,12 is a replica of problem 69 in [4]; the other two 2018-ChH problems will be published soon in the XXVII ChH booklet; the origin of 2018-WFNMC is lost in the past, but the given reasoning is genuine.

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