A Pedagogical Approach to Ramsey Multiplicity

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Abstract

It is well known that for all 2-colorings of the edges of K_6 there is a monochromatic triangle. Less well known is that there are two monochromatic triangles. More generally, for all 2-colorings of the edges of K_n there are roughly $\geq n^3/24$ monochromatic triangles. Another way to state this is that the density of monochromatic triangles is at least $1/4$.

The Ramsey Multiplicity of k is (asymptotically) the greatest α such that for every coloring of K_n the density of monochromatic K_k's is α. This concept has been studied for many years. We survey the area and provide proofs that are more complete, more motivated, and using modern notation.

1 Introduction

Throughout this paper we will let $n \in \mathbb{N}$ be a large natural number and $k \in \mathbb{N}$ be a small natural number (i.e. $k \ll n$). We are concerned with coloring the edges of K_n, the complete graph on n vertices. Let $c \in \mathbb{N}$ be the number of colors we use to do this. Our objective will be to find an asymptotic lower bound on the number of monochromatic copies of K_k in K_n for various values of k, n, and c.

Many, but not all, of the results in this paper are well established; however, some of the proofs in the literature are missing or incomplete. Many are not motivated. As such we present the proofs in a new light, intended to illuminate the problem solving process, while still rigorously proving the main results.

In Section 2 we show that for all colorings of the edges of K_6 there are two monochromatic K_3's. In Section 3 we use the ideas of the proof in Section 2 for all colorings of the edges of K_n, showing there are roughly at least $\frac{n^3}{24}$ monochromatic K_3's. We view this as saying that $\frac{1}{4}$ of the triangles are monochromatic.

What if we seek monochromatic K_k's? Section 4 introduces the concept of Ramsey Multiplicity, which is the fraction of K_k's that are the monochromatic. We discuss some of the early work, and give a lower bound on this fraction. In Section 5 we improve this lower bound. We offer two proofs of this improved lower bound. One is the proof from literature. The other is a motivated version of that proof. The motivated version is, in our opinion, easier to generalize to c

colors. We state the c-color version, though do not prove it, as the proof should be easy for the reader at that point.

In Section 6 we state the best known bounds. Finally, In Section 7, we discuss open questions.

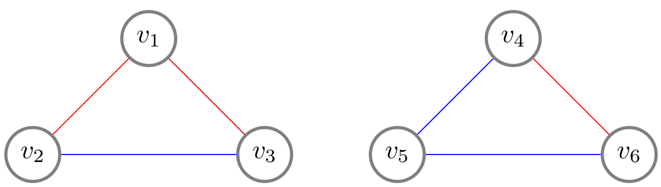
2 For all 2-Colorings of the Edges of K_6 … Triangles

Definition 1. We denote by $\psi_c(k,n)$ the minimum number of monochromatic K_k's in any c-coloring of K_n.

We give an example by showing that $\psi_2(3,6) \geq 2$. The proof is from an exposition by Dorwart & Finkbeiner [4] based on ideas from Schwenk [8].

Theorem 2. $\psi_2(3,6) \geq 2$

Proof. Define $COL : E \longrightarrow \{\text{red, blue}\}$ as a 2-coloring of the edges of our graph. Any triangle in our graph will either have 3 red edges, 3 blue edges, or it will be mixed with 2 edges of one color and 1 edge of the other. A mixed triangle would look like one of these:



Let R, B, and M be the sets of red, blue, and mixed triangles respectively. Then

$$|R| + |B| + |M| = \binom{6}{3} = 20$$

We show $|M| \leq 18$ which implies $|R| + |B| \geq 2$.

In each mixed triangle there will be exactly 2 vertices with both a red and blue edge coming out of them.

Definition 3. A Mix is an element $(v, \{u, w\}) \in V \times E$ s.t. $v \notin \{u, w\}$ and $COL(v, u) \neq COL(v, w)$. MIX is the set of all Mix's.

For example, in our mixed triangles above, the set of Mix's is:

$$\{(v_2, \{v_1, v_3\}), (v_3, \{v_1, v_2\}), (v_4, \{v_5, v_6\}), (v_6, \{v_4, v_5\})\}$$

Because there are exactly 2 Mix's for each mixed triangle, we see $|\text{MIX}| = 2|M|$. Now we bound $|\text{MIX}|$.

To bound the contribution of a single vertex, consider the red degree of each vertex in our graph, $d_R(v)$. Since every vertex has degree 5, $d_B(v) = 5 - d_R(v)$.

Case 1: $d_R(v) = 5$. Then v does not have different colored edges coming out of it so it contributes 0 to the count of our Mix's.

Case 2: $d_R(v) = 4$. Then $d_B(v) = 1$ and there are 4 pairs of edges of different colors coming out of v so this vertex contributes 4 to our count of MIX.

Case 3: $d_R(v) = 3$. Then $d_B(v) = 2$ and there are $3 \cdot 2 = 6$ pairs of edges of different colors coming out of v so this vertex contributes 6 to our count of MIX.

By symmetry we need not consider the cases $d_R(v) < 3$.

So each vertex in our graph will contribute at most 6 to the count of MIX. With 6 vertices in the graph this means

$$|\text{MIX}| \leq 6 \cdot 6 = 36 \Longrightarrow |M| \leq 18 \Longrightarrow |R| + |B| \geq 2$$

□

3 For all 2-Colorings of the Edges of $K_n \ldots$ Triangles

We rephrase Theorem 2:
For all 2-colorings of K_6 at least $\frac{2}{20} = \frac{1}{10}$ of the triangles are guaranteed to be monochromatic.

What happens if n is large? What should we expect the lower bound on the fraction of monochromatic triangles to be? We give an informal argument, credited to Erdős, for why the answer should be $\frac{1}{4}$ and then prove it formally.

Informal Argument Lower bounds on the Ramsey numbers are often obtained with the probabilistic method where a color is determined by a fair coin flip. Hence we assume that the coloring with the least number of monochromatic triangles is so determined. Color the edges of K_n as follows; for each edge, color it R with probability $\frac{1}{2}$, and (hence) B with probability $\frac{1}{2}$.

To get the density of triangles, pick three vertices at random. There are 8 possible ways to 2-color the edges of a triangle and 2 of those result in a monochromatic triangle. Hence the density of monochromatic triangles is $\frac{1}{4}$.
End of Informal Argument

In the proof of Theorem 2 the red and blue degrees of each vertex played a role in determining the maximum contribution to our set MIX. We saw the maximum contribution occurred when the red and blue degrees were close to equal. It will be useful to formalize this statement when considering coloring the edges of K_n. We leave the proof of this to the reader.

Lemma 4. *Let $x, y \in \mathbb{N}$. Then the maximum value of xy is achieved w.r.t. the constraint $x + y = d$ for some fixed $d \in \mathbb{N}$ when $x = \left\lfloor \frac{d}{2} \right\rfloor$ (or when $y = \left\lfloor \frac{d}{2} \right\rfloor$).*

We now prove a theorem about 2-coloring the edges of K_n. This result was first proved by Goodman [7] and later a simpler proof was given by Schwenk [8] based on the method of Theorem 2. Both authors demonstrate the upper and lower bounds agree in the leading order term. Here is a presentation of the proof by Schwenk, though we have modernized the notation of the original paper.

We split the theorem into two theorems: an upper bound and a lower bound.

Theorem 5. *For $n \geq 6$ a natural number,*

$$\psi_2(3,n) \geq \begin{cases} \frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}, & n \equiv 0 \pmod 2 \\ \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}, & n \equiv 1 \pmod 4 \\ \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} + \frac{1}{2}, & n \equiv 3 \pmod 4 \end{cases}$$

Proof. We proceed in a manner analogous to the previous example, constructing the sets R, B, M, and MIX, noting that $|\text{MIX}| = 2|M|$. To upper bound $|\text{MIX}|$ we consider the maximum number of mixed triangles that contain a vertex with two different colored edges coming out of it.

Case 1: $n \equiv 0 \mod 2$

The degree of each vertex $(n-1)$ is odd and therefore from Lemma 4, the maximum contribution of a given vertex to MIX is $\frac{n}{2} \cdot \frac{n-2}{2}$. So

$$\begin{aligned} |M| = \frac{|\text{MIX}|}{2} &\leq \frac{n^3 - 2n^2}{8} \\ &\Longrightarrow |R| + |B| \geq \binom{n}{3} - \frac{n^3 - 2n^2}{8} \\ &= \frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3} \end{aligned}$$

Case 2: $n \equiv 1 \mod 4$

Each vertex has degree $(n-1)$, which is an even number (divisible by 4). Using Lemma 4, this means we have a maximum contribution of $\frac{(n-1)^2}{4}$ to MIX from each vertex. Therefore

$$\begin{aligned} |M| = \frac{|\text{MIX}|}{2} &\leq n\frac{(n-1)^2}{8} \\ &\Longrightarrow |R| + |B| \geq \binom{n}{3} - n\frac{(n-1)^2}{8} \\ &= \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} \end{aligned}$$

Case 3: $n \equiv 3 \mod 4$

Since n is odd and $\frac{n-1}{2}$ is odd, our previous calculation of $|\text{MIX}|$ yields an odd number. We can't have half a triangle so we take $|M| = \left\lfloor \frac{|\text{MIX}|}{2} \right\rfloor =$

$\frac{|\text{MIX}|-1}{2}$.

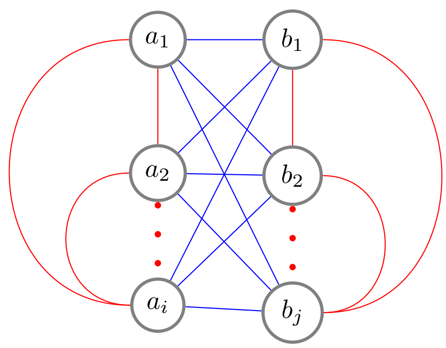
$$|M| = \frac{|\text{MIX}|-1}{2} \leq n\frac{(n-1)^2}{8} - \frac{1}{2}$$
$$\implies |R| + |B| \geq \binom{n}{3} - n\frac{(n-1)^2}{8} + \frac{1}{2}$$
$$= \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} + \frac{1}{2}$$

□

We now give the lower bound.

Theorem 6.
$$\psi_2(3,n) \leq \begin{cases} \frac{n^3}{24} - \frac{n^2}{8} + \frac{n}{12}, & n \text{ even} \\ \frac{n^3}{24} - \frac{n^2}{8} + \frac{5n}{24}, & n \text{ odd} \end{cases}$$

Proof. Our construction will proceed as follows. Divide the vertices into two equal sets, $A = \{a_i\}$ and $B = \{b_j\}$ (we consider two cases depending on the evenness of n). Color all the edges between vertices in the same set red and color all edges between vertices in different sets blue. Our picture looks like this:



We claim there are no blue triangles in this construction. Every set of 3 vertices must have either 2 vertices in one of the sets or 3 vertices in one of the sets. If all 3 vertices are in the same set, then we have a red triangle. If 2 vertices are in one of the sets, we have a red edge and again no blue triangle. Therefore we concern ourselves with counting the number of red triangles in our graph.

Case 1: n even

$|A| = |B| = \frac{n}{2}$. In each set there are thus $\binom{n/2}{3}$ red triangles.

$$2 \cdot \binom{\frac{n}{2}}{3} = \frac{n(n-1)(n-2)}{24} = \frac{n^3}{24} - \frac{n^2}{8} + \frac{n}{12}$$

Case 2: n odd

$|A| = \frac{n+1}{2}, |B| = \frac{n-1}{2}$.

$$\binom{\frac{n+1}{2}}{3} + \binom{\frac{n-1}{2}}{3} = \frac{(n+1)\cdot n \cdot (n-1)}{48} + \frac{(n-1)(n-2)(n-3)}{48}$$
$$= \frac{n^3}{48} - \frac{n}{48} + \frac{n^3}{48} - \frac{n^2}{8} + \frac{11n}{48} = \frac{n^3}{24} - \frac{n^2}{8} + \frac{5n}{24}$$

□

Corollary 7. *Let n be large.*

1. *For all two colorings of the edges of K_n, asymptotically at least $\frac{1}{4}$ of the triangles are monochromatic.*
2. *There exists a two-coloring of the edges of K_n where asymptotically at most $\frac{1}{4}$ of the triangles are monochromatic.*

BILL TO ROB: I decided to make these three unrelated points about triangles and K_n into one long note, hence being honest that these are just misc notes about the theorem. Proofread this passage and also fill in the FILL INs.

Note 8.

The upper and lower bounds on the number of monochromatic triangles do not match. The difference between them is:

1. $\frac{n^3}{24} - \frac{n^2}{8} + \frac{n}{12} - (\frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}) = \frac{n^2}{8} - \frac{n}{4}$ if $n \equiv 0 \pmod{2}$.
2. $\frac{n^3}{24} - \frac{n^2}{8} + \frac{5n}{12} - (\frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}) = \frac{n^2}{8} + \frac{5n}{24}$ if $n \equiv 1 \pmod{4}$.
3. $\frac{n^3}{24} - \frac{n^2}{8} + \frac{5n}{12} - (\frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} + \frac{1}{2}) = \frac{n^2}{8} + \frac{5n}{24} - \frac{1}{2}$ if $n \equiv 3 \pmod{4}$.

The construction we provide in Theorem 6 has the property that approximately half the edges are of each color. Erdős [5] conjectured this type of graph was the worst case for all colorings, which, as Corollary 7 shows, was true for $k = 3$. Is it true for general k? Alas no, Thomason [9] proved this was not the case for $k \geq 4$.

What about 3-colors? The following are known for large n:

1. Fox [6] showed that, for any 3-coloring of the edges of K_n, the fraction of triangles that are monochromatic is at least FILL IN PERHAPS WITH Ω NOTATION
2. Cummings et al. [3] showed that, for any 3-coloring of the edges of K_n, the fraction of triangles that are monochromatic is at FILL IN, PERHAPS WITH θ NOTATION.

4 Ramsey Multiplicity

Definition 9. The **Ramsey Multiplicity** given by:

$$RM_c(k) = \lim_{n\to\infty} \frac{\psi_c(k,n)}{\binom{n}{k}}$$

represents the minimum density of monochromatic subgraphs of size k in any c-coloring of K_n as n gets large.

Example 10. By Theorems 5 and 6, $RM_2(3) = \frac{1}{4}$.

That these limits exist for all c, k was first claimed by Erdős [5] without proof. Other authors have quoted it; however, to our knowledge, a proof has never been written down. We do so.

Lemma 11. *$\forall c, k, RM_c(k)$ exists and is finite.*

Proof. $RM_c(k) \leq 1, \forall c, k$ is clear. Now we must argue that the sequence is non-decreasing in n.

Consider a c-coloring of the graph $G = K_{n+1}$. Let G_v be the subgraph of G given by removing the vertex v (and its associated edges). Thus G_v is a c-coloring of K_n and as such will have at least $\psi_c(k,n)$ monochromatic K_k's. This is true for any subgraph G_v and therefore

$$\texttt{number of monochromatic } K_k\texttt{'s in } G_v \geq \psi_c(k,n)$$

There are $n+1$ choices for G_v and as such there are $(n+1)\psi_c(k,n)$ total monochromatic K_k's that can be counted in the associated subgraphs. We are counting some of these K_k multiple times and the question is how many times are we counting each? Each monochromatic K_k only appears once in a particular choice of G_v and the number of choices for v for which this K_k appears is $n+1-k$, choosing any vertex from our set of $n+1$ which is not part of the K_k. Therefore:

$$\psi_c(k, n+1) \geq \frac{n+1}{n+1-k}\psi_c(k,n)$$

Thus,

$$\begin{aligned} \frac{\psi_c(k,n+1)}{\binom{n+1}{k}} \Bigg/ \frac{\psi_c(k,n)}{\binom{n}{k}} &= \frac{(n+1-k)\psi_c(k,n+1)}{(n+1)\psi_c(k,n)} \\ &\geq \frac{(n+1)\psi_c(k,n)}{(n+1)\psi_c(k,n)} = 1 \end{aligned}$$

So our sequence is non-decreasing in n and upper bounded by 1 which means the limits exist $\forall c, k$. □

We now turn our attention to finding lower bounds on these constants for different values of c and k. First we must state a well-known bound on the Ramsey numbers.

Definition 12. $R_c(k)$ is defined to be the smallest natural number such that any c-coloring of a graph of this size will always have a monochromatic K_k.

Theorem 13. $R_2(k) \leq 4^k$.

Theorem 13 is folklore, and while there are better upper bounds (see Conlon [1]), we don't need to make use of them. With this result we can derive a loose bound on $RM_2(k)$. The proof of the following Theorem is due to Erdős [5]. We present it using modern notation and supply some of the missing details.

Theorem 14. $RM_2(k) \geq \left(\frac{1}{4}\right)^{k^2}$

Proof. Let $R = R_2(k)$. Let $\mathcal{A} = \{A_1, \cdots, A_{\binom{n}{R}}\}$ be an enumeration of all R-subsets of $[n]$. We will iterate the following process to find a lower bound on the number of monochromatic K_k's while $\mathcal{A}$ is not empty:

1. Choose $A_i \in \mathcal{A}$.
2. There is a monochromatic K_k in A_i, which increases our count. Call it C.
3. Remove from $\mathcal{A}$ every A_i containing C.

Every iteration produces a distinct monochromatic K_k and we are removing at most $\binom{n-k}{R-k}$ elements from $\mathcal{A}$. Hence:

$$\psi_2(k,n) \geq \frac{\binom{n}{R}}{\binom{n-k}{R-k}} = \frac{n!}{R!(n-R)!} \cdot \frac{(n-R)!(R-k)!}{(n-k)!} = \frac{n!}{(n-k)!} \cdot \frac{(R-k)!}{R!}$$

Now we take the lower bound $RM_2(3)$ by utilizing Theorem 13:

$$RM_2(k) = \lim_{n\to\infty} \frac{\psi_2(k,n)}{\binom{n}{k}} \geq \frac{\frac{n!}{(n-k)!} \cdot \frac{(R-k)!}{R!}}{\binom{n}{k}} = \frac{k!(R-k)!}{R!}$$
$$= \frac{1}{R} \cdot \frac{2}{R-1} \cdots \frac{k}{R-k+1} \geq \frac{1}{R^k} \geq \frac{1}{(4^k)^k} = \frac{1}{4^{k^2}}$$

∎

To get a sense of how this lower bound compares to known values, we utilize $RM_2(3) = \frac{1}{4}$ which we computed earlier. Theorem 14 gives $RM_2(3) \geq \frac{1}{4^{3^2}} = \frac{1}{262144}$, which is a significant disparity.

5 Comparing Two Proofs of a Tighter Bound

Now we wish to improve the bound from Theorem 14. The remainder of this section is based on the work of Conlon [2]. Theorem 17 is directly taken from Conlon's work as an example of a proof which greatly improves the bound in Theorem 14. Although the statement and proof are easy to understand, it is unclear how to generalize the result to c colors.

Notation 15. For the remaining Theorems we will use the notation $O_{a,b}(f(a,b,n))$ to denote that the function f has a coefficient of the highest-order term which depends on a and b.

Lemma 16. *Let $n \gg d$ and $0 < x < 1$ then*

$$\binom{x(n-1)}{d} \approx x^d \binom{n}{d}.$$

Proof. See Appendix. □

BILL TO ROB: PUT IN condition on a, b, n, perhaps $n > R_c(\max\{a,b\})2^{a+b}$ and note when used. .

Theorem 17. *Let $a, b \geq 1$ be natural numbers, assume n is large (i.e. $n \gg a$ and $n \gg b$). Then in any red/blue-coloring of the edges of K_n there are at least:*

$$2^{-a(b-2)-\binom{a+1}{2}}\binom{n}{a} - O_{a,b}(n^{a-1})$$

red K_a's OR at least:

$$2^{-b(a-2)-\binom{b+1}{2}}\binom{n}{b} - O_{a,b}(n^{b-1})$$

blue K_b's.

Proof. We induct on $a + b$.

Our base case is $a + b = 2$.

$$2^{-1(1-2)-\binom{2}{2}}\binom{n}{1} = n$$

and since we can think of every vertex being a monochromatic K_1 in either color, the statement holds.

Now we assume our theorem holds for $a_0 < a, b_0 = b$ and we prove it is true at a, b.

For each vertex v_i there is a color C_i s.t. there are at least $\frac{n-1}{2}$ neighbors to which it is connected by C_i. In our entire graph there must be at least $\frac{n}{2}$ of these vertices associated with either red or blue. Suppose WLOG these vertices are red and we call them $\{v_1, \cdots, v_{\frac{n}{2}}\}$. Let V_i be the respective red neighbors of $v_i, \forall i$. For each V_i, our induction hypothesis tells us there are either:

$$2^{-(a-1)(b-2)-\binom{a}{2}}\binom{\frac{n-1}{2}}{a-1} - O_{a,b}((n/2)^{a-2})$$

red K_{a-1}'s OR at least:

$$2^{-(b)(a-3)-\binom{b+1}{2}}\binom{\frac{n-1}{2}}{b} - O_{a,b}((n/2)^{b-1})$$

blue K_b's.

First suppose there exists i for which the latter case holds for V_i. Then the number of blue K_b's is at least:

$$\begin{aligned}
&2^{-(b)(a-3)-\binom{b+1}{2}}\binom{\frac{n-1}{2}}{b} - O_{a,b}((n/2)^{b-1}) \\
&= 2^{-(b)(a-3)-\binom{b+1}{2}}2^{-b}\binom{n}{b} - O_{a,b}(n^{b-1}) \\
&= 2^{-(b)(a-2)-\binom{b+1}{2}}\binom{n}{b} - O_{a,b}(n^{b-1})
\end{aligned}$$

completing the proof. Therefore we assume for all i, V_i instead has at least:

$$2^{-(a-1)(b-2)-\binom{a}{2}}\binom{\frac{n-1}{2}}{a-1} - O_{a,b}((n/2)^{a-2})$$

red K_{a-1}'s. By our assumption, since each V_i is connected to v_i by red edges, each of these forms a red K_a. It is possible we have counted each of these a times (once per each vertex), and there are at least $\frac{n}{2}$ of them so in total we have at least:

$$\begin{aligned}
&\frac{1}{a}\cdot\frac{n}{2}\left(2^{-(a-1)(b-2)-\binom{a}{2}}\binom{\frac{n-1}{2}}{a-1} - O_{a,b}((n/2)^{a-2})\right) \\
&= \frac{n}{2a}\left(2^{-(a-1)(b-2)-\binom{a}{2}}2^{1-a}\binom{n}{a-1} - O_{a,b}(n^{a-2})\right) \\
&= 2^{-(a-1)(b-2)-\binom{a}{2}}2^{-a}\binom{n}{a} - O_{a,b}(n^{a-1}) \\
&= 2^{-(a-1)(b-2)-\binom{a+1}{2}}\binom{n}{a} - O_{a,b}(n^{a-1}) \\
&> 2^{-(a)(b-2)-\binom{a+1}{2}}\binom{n}{a} - O_{a,b}(n^{a-1})
\end{aligned}$$

red K_a's. □

Corollary 18. $RM_2(k) \geq \left(\frac{1}{2\sqrt{2}}\right)^{k^2(1-o(1))}$

Proof. In the previous theorem, let $a = b = k$. Then

$$\begin{aligned}
\psi_2(k,n) &\geq 2^{-k(k-2)-\binom{k+1}{2}}\binom{n}{k} - O_k(n^{k-1}) \\
&= 2^{-\frac{3}{2}k^2+\frac{3}{2}k}\binom{n}{k} - O_k(n^{k-1}) \\
&= \left(\frac{1}{2\sqrt{2}}\right)^{k^2-k}\binom{n}{k} - O_k(n^{k-1})
\end{aligned}$$

So

$$RM_2(k) = \lim_{n\to\infty} \frac{\psi_2(k,n)}{\binom{n}{k}} \geq \left(\frac{1}{2\sqrt{2}}\right)^{k^2(1-o(1))}$$

□

Theorem 17 is succinct and the proof is easy to follow. A curious reader may wonder where the leading coefficients came from and we must postpone this exploration until our next result. As a comparison with the bound we computed for $RM_2(3)$ earlier, this result gives us:

$$RM_2(3) \geq \left(\frac{1}{2\sqrt{2}}\right)^{3^2} \approx \frac{1}{11585}$$

We now step back and consider the problem in generality. Due to the definition of Ramsey Multiplicity, we want to construct a function whose leading term is a product of some coefficient with $\binom{n}{k}$. If we structure the Theorem in a similar way, this function can depend on both a and b to represent different sized subgraphs for each color. The lower order terms are of no consequence as we plan to take a limit. With this in mind, we state and prove a similar result to Theorem 17 as a second method of obtaining the bound in Corollary 18.

Rather than simply state the Theorem in its entirety upfront, we methodically proceed from our general statement and derive relations on our functions. Then, under certain intuitive, relaxed conditions, we use these relations to provide a recurrence for our functions. Finally we solve the recurrence and realize the same bound from Corollary 18.

Theorem 19. *Let $a, b \geq 1$ be natural numbers and assume n is large (i.e. $n \gg a$ and $n \gg b$). Let $T = T(a,b)$ and $U = U(a,b)$ be functions (determined later). Then in any red/blue-coloring of the edges of K_n there are at least:*

$$T\binom{n}{a} - O_{a,b}(n^{a-1})$$

red K_a's OR at least:

$$U\binom{n}{b} - O_{a,b}(n^{b-1})$$

blue K_b's.

Proof. We wish to prove this by induction, in a similar manner to the previous Theorem, but because we will later determine T and U, we will only get a set of relations for these functions in order for our inductive proof to work.

Again we induct on $a+b$ and our base case is $a+b=2$. Note that if $a=1$ or $b=1$ we have n monochromatic K_a's or K_b's respectively. So we can simply set:

$$T(a,1) = T(1,b) = U(a,1) = U(1,b) = 1, \forall a, b \tag{1}$$

Now assuming our theorem holds for $a_0 < a$, or $b_0 < b$, we wish to prove it holds at a, b.

Every vertex v is connected to either $\frac{n-1}{2}$ vertices by blue edges or $\frac{n-1}{2}$ vertices by red edges. Additionally, either the first case occurs $\frac{n}{2}$ times or the second case occurs $\frac{n}{2}$ times. We will first work through the case where we have $\frac{n}{2}$ vertices $\{v_i\}$ each connected to $\frac{n-1}{2}$ vertices (respectively V_i) by blue edges and the remaining cases follow similarly. Our induction hypothesis says for each V_i we must have at least:

$$T(a, b-1)\binom{\frac{n-1}{2}}{a} - O_{a,b}(n^{a-1})$$

red K_a's OR at least:

$$U(a, b-1)\binom{\frac{n-1}{2}}{b-1} - O_{a,b}(n^{b-2})$$

blue K_{b-1}'s.

If the first case occurs for any of the vertices, then (making use of Lemma 16), we have at least:

$$T(a, b-1)\binom{\frac{n-1}{2}}{a} - O_{a,b}(n^{a-1}) = \left(\frac{1}{2}\right)^a T(a, b-1)\binom{n}{a} - O_{a,b}(n^{a-1})$$

red K_a's. To complete our proof from here we would need the following relation on T:

$$\left(\frac{1}{2}\right)^a T(a, b-1) \geq T(a, b) \tag{2}$$

Now if this case does not occur for any of the vertices v_i, then the second case occurs for every v_i and because there are at least $\frac{n}{2}$ vertices which we may be overcounting b times, once for every vertex of each K_b, we have at least:

$$\begin{aligned}
&\frac{1}{b} \cdot \frac{n}{2}\left(U(a, b-1)\binom{\frac{n-1}{2}}{b-1} - O_{a,b}(n^{b-2})\right) \\
&= \frac{1}{b} \cdot \frac{n}{2}\left(U(a, b-1)\left(\frac{1}{2}\right)^{b-1}\binom{n}{b-1} - O_{a,b}(n^{b-2})\right) \\
&= \left(\frac{1}{2}\right)^b U(a, b-1)\binom{n}{b} - O_{a,b}(n^{b-1})
\end{aligned}$$

blue K_b's. To complete our proof from here we would need the following relation on U:

$$\left(\frac{1}{2}\right)^b U(a, b-1) \geq U(a, b) \tag{3}$$

If we worked through the other cases, we would similarly obtain the following relations on T and U:

$$\left(\frac{1}{2}\right)^a T(a,b-1) \geq T(a,b)$$
$$\left(\frac{1}{2}\right)^b U(a,b-1) \geq U(a,b)$$
$$\left(\frac{1}{2}\right)^a T(a-1,b) \geq T(a,b)$$
$$\left(\frac{1}{2}\right)^b U(a-1,b) \geq U(a,b)$$
$$\left(\frac{1}{2}\right)^a T(a-1,b) \geq T(a,b)$$
$$\left(\frac{1}{2}\right)^b U(a-1,b) \geq U(a,b)$$

Note the redundancy in some of these equations. This is actually due to an arbitrary, although understandable choice we made earlier in the proof, and now is a good time for an aside to look at this in a bit more detail.

Earlier we said "Every vertex v is connected to either $\frac{n-1}{2}$ *vertices* $\cdots$*" This is true, but our choice of the fraction* $\frac{1}{2}$ *may not have been optimal. Conlon indeed shows that this constant is not the optimal choice for analysis and instead letting this value vary depending on the values of a and b is ideal. We can offer some intuition as to why this might be the case. Consider the scenario where b is much larger than a, say* $b = 100a$*. In this scenario we could imagine the worst-case graph may be one with significantly more blue edges than red edges, and could thus imagine tweaking our proof to account for this possibility. Still, our choice of* $\frac{1}{2}$ *will allow us to determine a non-trivial bound as desired and it greatly reduces the strain on our computations so we will now proceed with this in mind.*

For clarity we will now list our required relations on T and U to complete the proof.

$$T(a,1) = U(1,b) = 1, \forall a,b \tag{4}$$
$$\left(\frac{1}{2}\right)^a T(a,b-1) \geq T(a,b) \tag{5}$$
$$\left(\frac{1}{2}\right)^b U(a,b-1) \geq U(a,b) \tag{6}$$
$$\left(\frac{1}{2}\right)^a T(a-1,b) \geq T(a,b) \tag{7}$$
$$\left(\frac{1}{2}\right)^b U(a-1,b) \geq U(a,b) \tag{8}$$

□

In Section 6 we state a result of Conlon who provides a numerical solution to this recurrence, although the methods used are beyond the scope of this paper. Now we proceed to show how to obtain the same bound referenced earlier using these relations.

Corollary 20. $RM_2(k) \geq \left(\frac{1}{2\sqrt{2}}\right)^{k^2(1-o(1))}$

Proof. We wish to maximize the quantity $T(k,k)$, noting that the relations are symmetric in T and U when $a = b = k$.

To do this, imagine a lattice of points $\{1 \leq a \leq k\} \times \{1 \leq b \leq k\}$. To reach the point (k,k) we must begin along an edge and sequentially take steps, increasing by one our 1^{st} or 2^{nd} coordinate. Each of these steps comes at a multiplicative cost of 2^{-a} or 2^{-b} in our respective coordinates. Importantly, because $T(a,b)$ must be smaller than the product of each step, we need to minimize the quantity over all possible step sequences. Let us formalize this process with a definition:

Definition 21. Let $[x] \times [y]$ be an integer lattice of points. A **path** P in this lattice to (x,y) is defined as a sequence of points $\{a_i, b_i\}_{i=0,\cdots,t}$ which satisfies the following properties:

1. $a_0 = 1$ or $b_0 = 1$
2. $a_1 \neq 1$ and $b_1 \neq 1$
3. $(a_i, b_i) = (a_{i-1}+1, b_{i-1})$ or $(a_i, b_i) = (a_{i-1}, b_{i-1}+1)$
4. $(a_t, b_t) = (x, y)$

We will denote by $\mathcal{P}_{x,y}$ the set of all paths to (x,y).

Using this definition and notation we can succinctly describe our optimization problem as:

$$T(k,k) = \min_{P \in \mathcal{P}_{k,k}} \Pi_{i=0}^{t} 2^{-a_i}$$

To minimize this product, we wish to maximize the exponents $\{a_i\}$ as quickly as possible and this can be done with the path (2, 1), (2, 2), (3, 2), (4, 2), $\cdots$, (k, 2), (k, 3), $\cdots$, (k, k).

$$\begin{aligned}
T(k,k) &= \min_{P \in \mathcal{P}_{k,k}} \Pi_{i=1}^{t} 2^{-a_i} \\
&= 2^{-(2+\sum_{i=2}^{k} i + \sum_{i=2}^{k} k)} \\
&= 2^{-(2-1+(k(k+1)/2)+k(k-1))} \\
&= 2^{-(\frac{3}{2}k^2 - \frac{1}{2}k + 1)} \\
&= \left(\frac{1}{2\sqrt{2}}\right)^{k^2(1-o(1))}
\end{aligned}$$

□

Conlon states the process used to obtain the bounds for 2-colorings is also effective for c-colorings, but does not explore this in his original work. We now offer our main result, a generalization of the previous Theorems for c-colorings. The technique used to solve the problem will be exactly as presented in the proof of Theorem 19 and the subsequent Corollary 18.

Theorem 22. *Let $c \geq 2$ represent the number of colors, $\boldsymbol{m} = \{m_1, \cdots, m_c\}$ with $m_i \geq 1\, \forall i$ and assume n is large. Let $\{U_i = U_i(\boldsymbol{m})\}_{i=1}^{c}$ be functions (determined later). Then in any c-coloring of the edges of K_n there are at least:*

$$U_1 \binom{n}{m_1} - O_{\boldsymbol{m}}(n^{m_1 - 1})$$

monochromatic K_{m_1}'s OR at least:

$$\vdots$$

$$U_c \binom{n}{m_c} - O_{\boldsymbol{m}}(n^{m_c - 1})$$

monochromatic K_{m_c}'s.

Proof. We prove this by induction on $\sum_i m_i$.

Base case: Any $m_i = 1$. Then we have n mono K_{m_i}'s. In this case we may set $U = 1$ if there is a 1 in any coordinate of the vector $\mathbf{m}$.

Define $\mathbf{m}_0 = (m_{1_0}, \cdots, m_{c_0})$.

Our inductive hypothesis is that the theorem holds for any $\mathbf{s}$ with $s_i \leq m_{i_0}, \forall i$ and $\sum_i s_i < \sum_i m_{i_0}$. We wish to prove it is true at equality.

For each vertex v there must be a color C_i for which there are at least $\frac{n-1}{c}$ edges of this color connecting v to other vertices. Across all vertices, choose the color C_i which occurs the most often, which results in a set $\{v_i\}$ of size at least $\frac{n}{c}$. WLOG suppose this color is C_1. Let V_i be the set of vertices connected to each v_i by color C_1.

By our inductive hypothesis this means there are at least:

$$U_1(m_1 - 1, m_2, \cdots, m_c) \binom{\frac{n-1}{c}}{m_1 - 1} - O_{\mathbf{m}}(n^{m_1 - 2})$$

monochromatic K_{m_1}'s of color C_1 OR at least:

$$U_2(m_1 - 1, m_2, \cdots, m_c) \binom{\frac{n-1}{c}}{m_2} - O_{\mathbf{m}}(n^{m_2 - 1})$$

monochromatic K_{m_2}'s of color C_2 OR at least:

$$U_3(m_1 - 1, m_2, \cdots, m_c) \binom{\frac{n-1}{c}}{m_3} - O_{\mathbf{m}}(n^{m_3 - 1})$$

monochromatic K_{m_3}'s of color C_3 OR at least:

$$\vdots$$

$$U_c(m_1-1,m_2,\cdots,m_c)\binom{\frac{n-1}{c}}{m_c} - O_{\mathbf{m}}(n^{m_c-1})$$

monochromatic K_{m_c}'s of color C_c.

For $2 \le j \le c$, our argument proceeds in the following way.

$$U_j(m_1-1,m_2,\cdots,m_c)\binom{\frac{n-1}{c}}{m_j} - O_{\mathbf{m}}(n^{m_j-1}) = \left(\frac{1}{c}\right)^{m_j} U_j(m_1-1,m_2,\cdots,m_c)\binom{n}{m_j} - O_{\mathbf{m}}(n^{m_j-1})$$

$$\Longrightarrow \left(\frac{1}{c}\right)^{m_j} U_j(m_1-1,m_2,\cdots,m_c) \ge U_j(m_1,m_2,\cdots,m_c)$$

If none of these cases happens, then all of our vertices are connected to each of their corresponding vertex sets by the same color, C_1. We have one monochromatic K_{m_1} for each of these vertices, but we may be overcounting the total by a factor of m_1, so:

$$\begin{aligned}
&\frac{n}{c}\cdot\frac{1}{m_1}\left(U_1(m_1-1,m_2,\cdots,m_c)\binom{\frac{n-1}{c}}{m_1-1} - O_{\mathbf{m}}(n^{m_1-2})\right)\\
&= \left(\frac{1}{c}\right)^{m_1} U_1(m_1-1,m_2,\cdots,m_c)\binom{n}{m_1} - O_{\mathbf{m}}(n^{m_1-1})\\
&\Longrightarrow \left(\frac{1}{c}\right)^{m_1} U_1(m_1-1,m_2,\cdots,m_c) \ge U_1(m_1,m_2,\cdots,m_c)
\end{aligned}$$

The relations we desire on $\{U\}$ must hold for each color and each coordinate of our vector $\mathbf{m}$, so we are left with the following set of relations that must be satisfied for our Theorem to hold:

$$U(m_{1_0},\cdots,m_{i_0}=1,\cdots,m_{c_0}) = 1, 1 \le i \le c, 1 \le m_{j_0} \le m_j, \forall i,j \tag{9}$$

$$c^{-\max_i\{m_{i_0}\}}U(m_{1_0},\cdots,m_{i_0}-1,\cdots,m_{c_0}) \ge U(m_{1_0},\cdots,m_{c_0}), \forall i \tag{10}$$

□

Corollary 23. $RM_c(k) \ge \left(\left(\frac{1}{c}\right)^{c-\frac{1}{2}}\right)^{k^2(1-o(1))}$

Proof. Similar to our construction of the function T in the proof of Theorem 19 we wish to find paths, now in a c-dimensional lattice. The minimizing path to reach $(k,\cdots,k)$ can be done with the path (2, $\cdots$, 2, 1), (2, $\cdots$, 2, 2), (3, $\cdots$, 2, 2), (4, $\cdots$, 2, 2), $\cdots$, (k, $\cdots$, 2, 2), (k, 3, $\cdots$, 2, 2), $\cdots$, (k, k, 2, $\cdots$, 2, 2), $\cdots$, (k, k, $\cdots$, k, 2), $\cdots$, (k, $\cdots$, k). Thus

$$\begin{aligned}
U(k,\cdots,k) &= \min_{P\in\mathcal{P}_{k,\cdots,k}} \Pi_{i=1}^t c^{-a_i}\\
&= c^{-(2+\sum_{i=2}^k i+(c-1)\sum_{i=2}^k k)}\\
&= c^{-((2-1+k(k+1)/2)+(c-1)k(k-1))}\\
&= c^{-((c-\frac{1}{2})k^2-(c-\frac{3}{2})k+1)}\\
&= \left(\left(\frac{1}{c}\right)^{c-\frac{1}{2}}\right)^{k^2(1-o(1))}
\end{aligned}$$

□

Corollary 24. $RM_3(k) \geq \left(\frac{1}{9\sqrt{3}}\right)^{k^2(1-o(1))}$

6 Tighter Bounds

Conlon provides an analytic approximation to the solution of the recurrence given in Equations 4 - 8. Keep in mind that our arbitrary choice of the fraction $\frac{1}{2}$ was not optimal and their work utilizes a more general recurrence.

Theorem 25. *Let $t_\epsilon(x)$ be a function with $t_\epsilon(0) = \epsilon$ and satisfying the differential equation:*

$$t'_\epsilon(x) = \log t_\epsilon(x) \frac{t_\epsilon(x)(1 - t_\epsilon(x))}{x - (1-x)t_\epsilon(x)}$$

Let $L = \lim_{\epsilon \to 0} t_\epsilon(1)$ and $C = (L(1-L))^{-1/2}$, then

$$RM_2(k) \geq C^{-k^2(1-o(1)}$$

A numeric approximation yields the value $C \approx 2.18$. This is the best known bound and we therefore have the following result:

$$RM_2(3) \geq \left(\frac{1}{2.18}\right)^{3^2} \approx \frac{1}{1112}$$

Their analysis is simple enough to follow, but goes beyond the scope of this paper. From a quick glance it is not immediately obvious to us how their process could be generalized for c-colorings, but that is an area for further research.

7 Open Problems

BILL TO ROB: I rewrote the open problems. Proofread and correct. Add some if you are so inclined.

Open 26.

1. *The known upper and lower bounds on $\phi_2(3, n)$ (Theorems 5 and 6) differ by a quadratic term (see the note following Corollary 7. Improve this gap. Perhaps gather empirical evidence.*
2. *The results on $\phi_2(3, n)$ are obtained with completely elementary techniques. Can this be done for $\phi_2(4, n)$? $\phi_3(3, n)$?, $\phi_2(5, n)$?*
3. *Obtain an easier proof of Theorem 25. One litmus test is if the proof easily generalizes to c colors.*

References

[1] David Conlon. A new upper bound for diagonal Ramsey numbers. *Annals of Mathematics*, 170(2):941–960, 2009. `https://arxiv.org/abs/math/0607788`.

[2] David Conlon. On the Ramsey multiplicity of complete graphs. *Combinatorica*, 32:171–186, 2012. `https://arxiv.org/abs/0711.4999`.

[3] James Cummings, Daniel Král, Florian Pfender, Konrad Sperfeld, Andrew Treglown, and Michael Young. Monochromatic triangles in three-coloured graphs. *Journal of Combinatorial Theory, Series B*, 103(4):489–503, 2013.

[4] H.L. Dorwart and T.D. Finkbeiner. *Mathematical Plums*, chapter Chromatic Graphs. MAA, 1979. Edited by R. Honsberger.

[5] Paul Erdős. On the number of complete subgraphs contained in certain graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl*, 7(3):459–464, 1962.

[6] Jacob Fox. There exists graphs with super-exponential Ramsey multiplicity constant. *Journal of Graph Theory*, 57:89–98, 2008.

[7] A. W. Goodman. On sets of acquaintances and strangers at any party. *The American Mathematical Monthly*, 66(9):778–783, 1959.

[8] A. J. Schwenk. Acquaintance graph party problem. *The American Mathematical Monthly*, 79(10):1113–1117, 1972.

[9] Andrew Thomason. A Disproof of a Conjecture of Erdős in Ramsey Theory. *Journal of the London Mathematical Society*, s2-39(2):246–255, 04 1989.

A Appendix

Proof of Lemma 16

Proof. We wish to show for n large, $d \ll n$, and $0 < x < 1$ fixed:

$$\binom{x(n-1)}{d} = x^d \binom{n}{d} - O(n^{d-1})$$

We carefully keep track of the sign of the lower order term for completeness, though this is not strictly necessary for our earlier results because we are taking limits as $n \longrightarrow \infty$.

$$
\begin{aligned}
\binom{x(n-1)}{d} &= \frac{x(n-1)\cdot(x(n-1)-1)\cdots(x(n-1)-d+1)}{d!} \\
&= \frac{(x(n-1))^d - \left[\sum_{i=1}^{d-1} i\right](x(n-1))^{d-1}}{d!} + O(n^{d-2}) \\
&= x^d\frac{(n-1)^d}{d!} - x^{d-1}\frac{\left[\sum_{i=1}^{d-1} i\right](n-1)^{d-1}}{d!} + O(n^{d-2}) \\
&= x^d\frac{n^d}{d!} - x^d\frac{(d-1)(n-1)^{d-1}}{d!} - x^{d-1}\frac{\left[\sum_{i=1}^{d-1} i\right](n-1)^{d-1}}{d!} + O(n^{d-2}) \\
&= x^d\binom{n}{d} - x^d\binom{n}{d} + x^d\frac{n^d}{d!} - x^d\frac{(d-1)(n-1)^{d-1}}{d!} - x^{d-1}\frac{\left[\sum_{i=1}^{d-1} i\right](n-1)^{d-1}}{d!} + O(n^{d-2}) \\
&= x^d\binom{n}{d} - x^d\frac{n^d}{d!} + x^d\frac{\left[\sum_{i=1}^{d-1} i\right]n^{d-1}}{d!} + x^d\frac{n^d}{d!} \\
&\qquad - x^d\frac{(d-1)(n-1)^{d-1}}{d!} - x^{d-1}\frac{\left[\sum_{i=1}^{d-1} i\right](n-1)^{d-1}}{d!} + O(n^{d-2}) \\
&= x^d\binom{n}{d} + x^d\frac{\left[\sum_{i=1}^{d-1} i\right]n^{d-1}}{d!} - x^d\frac{(d-1)(n-1)^{d-1}}{d!} - x^{d-1}\frac{\left[\sum_{i=1}^{d-1} i\right](n-1)^{d-1}}{d!} + O(n^{d-2})
\end{aligned}
$$

Comparing the coefficients of n^{d-1} in the second and fourth terms of this last equation, since $x < 1$ we see that of the fourth term is larger in absolute value and thus:

$$\binom{x(n-1)}{d} = x^d\binom{n}{d} - O(n^{d-1})$$

□