The Algebraic Degree of cos($\frac{a\pi}{m}$)
by
William Gasarch
Auguste Gezalyan

1 Introduction

The following are well known: (a) $\cos(\pi/1) = -1$, (b) $\cos(\pi/2) = 0$, (c) $\cos(\pi/3) = \frac{1}{2}$, (d) $\cos(\pi/4) = \frac{\sqrt{2}}{2}$, and (e) $\cos(\pi/6) = \frac{\sqrt{3}}{2}$. Note that $\cos(\pi/5)$ is missing. In Harold Boas's paper [1] he shows that $\cos(\pi/5) = \frac{1+\sqrt{5}}{4}$, which is half the golden ratio. Note that all of these numbers are algebraic.

Are numbers of the form $\cos(a\pi/b)$ with $a, b \in \mathsf{N}$ always algebraic? Yes. This is well known. In this paper we prove the theorem with an eye towards (a) getting explicit polynomials, and (b) seeing what the degree of those polynomials are.

To state our results we need two definitions.

Def 1.1 Let $m \in \mathsf{N}$.

1. If m is odd then

$$f_{\rm odd}(m) = \left|\left\{k \colon \left(1 \le k \le \frac{m-1}{2}\right) \wedge \gcd(k,m) = 1\right\}\right|.$$

   $f_{\rm odd}(m)$ is the number of numbers in $\{1, \ldots, \frac{m-1}{2}\}$ that

2. If m is even then

$$f_{\rm even}(m) = |\{k \colon (1 \le k \le m) \wedge \gcd(k,m) = 1\}|.$$

   $f_{\rm even}(m)$ is the number of numbers in $\{1, \ldots, m\}$ that are relatively prime to m. Note that $f_{\rm even}(m)$ is $\phi(m)$ where ϕ is the Euler Totient function.

In Sections 2 and 3 we establish when $\cos(a\pi/b) = \cos(na\pi/b)$. This is needed for later sections. In Section 4 we show that, for all $k, m \in \mathsf{N}$, $m \ge 3$, m odd, $1 \le k \le m-1$, and $\gcd(k,m) = 1$, $\cos(\frac{k\pi}{m})$ is algebraic of degree $\le f_{\rm odd}(m)$. In Section 5 we show that, for all $k, m \in \mathsf{N}$, $m \ge 3$, m even, $1 \le k \le m-1$, and $\gcd(k,m) = 1$, $\cos(\frac{k\pi}{m})$ is algebraic of degree $\le f_{\rm even}(m)$. In the appendix we list out many of the polynomials show to exist in Sections 4 and 5.

Clearly from these results one can obtain the $\sin(a\pi/b)$ is algebraic. Can one obtain this directly, which might lead to lower degrees? In Section 6 we show why this is unlikely.

2 Chebyshev Polynomials of the First Kind

Def 2.1 The Chebyshev Polynomials of the first kind are, for all $n \in \mathsf{N}$,

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2-1)^k x^{n-2k} \text{ [2]}$$

The following theorem is well known.

Theorem 2.2 *For all $n \in \mathsf{N}$, $T_n(\cos(\theta)) = \cos(n\theta)$.*

3 When is $\cos(\frac{a\pi}{b}) = \cos(\frac{na\pi}{b})$?

Lemma 3.1

1. *Let $n \in \mathsf{N}$. For all*

$$\theta \in \left\{ \frac{2k\pi}{n-1} : k \in \mathsf{Z}, n > 1 \right\} \bigcup \left\{ \frac{2k\pi}{n+1} : k \in \mathsf{Z} \right\}$$

$\cos(\theta) = \cos(n\theta)$.

2. *If n is odd then the n roots of $T_n(x) - x = 0$ are*

$$\left\{ \cos\left(\frac{2k\pi}{n-1}\right) : 0 \le k \le \frac{n-1}{2} \right\} \bigcup \left\{ \cos\left(\frac{2k\pi}{n+1}\right) : 1 \le k \le \frac{n-1}{2} \right\}.$$

(If $n = 1$ then just take the second union.)

3. *If n is even then the n roots of $T_n(x) - x = 0$ are*

$$\left\{ \cos\left(\frac{2k\pi}{n-1}\right) : 0 \le k \le \frac{n-2}{2} \right\} \bigcup \left\{ \cos\left(\frac{2k\pi}{n+1}\right) : 1 \le k \le \frac{n}{2} \right\}.$$

(If $n = 1$ then just take the second union.)

Proof:
1) For the first union notice that

$$\cos\left(\frac{2k\pi}{n-1}\right) = \cos\left(\frac{2k\pi}{n-1} + 2k\pi\right) = \cos\left(\frac{n2k\pi}{n-1}\right).$$

For the second union notice that

$$\cos\left(\frac{2k\pi}{n+1}\right) = \cos\left(-\frac{2k\pi}{n+1}\right) = \cos\left(2\pi k - \frac{2k\pi}{n+1}\right) = \cos\left(\frac{n2k\pi}{n+1}\right).$$

2) By Theorem 2.2 and Part 1 we have that all of the elements in

$$X = \left\{\cos\left(\frac{2k\pi}{n-1}\right) : 0 \le k \le \frac{n-1}{2}\right\} \bigcup \left\{\cos\left(\frac{2k\pi}{n+1}\right) : 1 \le k \le \frac{n-1}{2}\right\}$$

are roots of $T_n(x) - x = 0$. By algebra one can see that all of the angles mentioned are distinct and in $[0, \pi]$. Since cosine is injective on $[0, \pi]$, X has n different numbers. Since $T_n(x) - x$ is of degree n, the elements of X are its n roots.

3) Similar to the proof of Part 2. ▮

4 A Polynomial for $\cos(\frac{k\pi}{m})$: m Odd

Lemma 4.1 *Let $m \in \mathsf{N}$, $m \ge 3$, m odd. Let*

$$A_{m,1} = \left\{\cos\left(\frac{2k\pi}{m}\right) : \left(1 \le k \le \frac{m-1}{2}\right) \wedge \gcd(k, m) = 1\right\}$$

Then $A_{m,1}$ is a subset of the roots of $T_{m-1}(x) - x$.

Proof: Let $n = m - 1$. Then n is even. Note that we can write $A_{m,1}$ as

$$\left\{\cos\left(\frac{2k\pi}{n+1}\right) : \left(1 \le k \le \frac{n}{2}\right) \wedge \gcd(k, n+1) = 1\right\}$$

This is a subset of

$$\left\{\cos\left(\frac{2k\pi}{n+1}\right) : \left(1 \le k \le \frac{n}{2}\right)\right\}$$

By Lemma 3.1 this is a subset of the roots of $T_n(x) - x$ which is $T_{m-1}(x) - x$. ▮

Theorem 4.2 *Let $m \in \mathsf{N}$, $m \ge 3$, m odd.*

1. *There exists $p_m(x) \in \mathsf{Z}[x]$ of degree $f_{\mathrm{odd}}(m)$ whose roots are*

$$A_{m,1} = \left\{\cos\left(\frac{2k\pi}{m}\right) : \left(1 \le k \le \frac{m-1}{2}\right) \wedge \gcd(k, m) = 1\right\}$$

2. *All of the elements of $A_{m,1}$ are algebraic of degree $\le f_{\mathrm{odd}}(m)$. (This follows from Part 1.)*

3. *There exists $q_m(x) \in \mathsf{Z}[x]$ of degree $f_{\mathrm{odd}}(m)$ whose roots are*

$$A_{m,2} = \left\{\cos\left(\frac{(m-2k)\pi}{m}\right) : \left(1 \le k \le \frac{m-1}{2}\right) \wedge \gcd(k, m) = 1\right\}$$

4. *All of the elements of $A_{m,2}$ are algebraic of degree $\leq f_{\text{odd}}(m)$. (This follows from Part 3.)*

5. *All of the elements of*

$$\left\{\cos\left(\frac{k\pi}{m}\right) : \left(1 \leq k \leq m-1\right) \wedge \gcd(k,m) = 1\right\}$$

algebraic of degree $\leq f_{\text{odd}}(m)$. (This follows from Parts 2 and 4.)

Proof: To construct $p_m(x)$ we will implement the following strategy: We begin by constructing a fraction with $T_{m-1}(x) - x$ as the numerator. Note that the roots of $T_{m-1}(x) - x$ include the roots we want for $p_m(x)$. By Lemma 4.1 the roots of $p_m(x)$ are a subset of the roots of $T_{m-1}(x) - x$. In order to remove the other roots of $T_{m-1}(x) - x$ we will strategically place previously constructed polynomials, $p_{m'}(x)$, $m' < m$ in the denominator. Since all the $p_{m'}(x)$ divide $T_{m-1}(x) - x$ and have unique roots, in the end we will be left with a polynomial with only our desired roots. Parts 2 and 3 of the above theorem will follow naturally from the construction of $p_m(x)$.

(1) We prove Part 1 by induction on m.

Base Case: $m = 3$. Then $A_{m,1} = \{\cos(2\pi/3)\} = \{-1/2\}$ and $A_{m,2} = \{\cos(\pi/3)\} = \{1/2\}$. Let $p_3(x) = 2x + 1$ and $q_3(x) = -2x + 1$. Both $p_3(x)$ and $q_3(x)$ are of degree $f_{\text{odd}}(3) = 1$.

Inductive Hypothesis The theorem is true for all odd m', $3 \leq m' < m$.

Inductive Step Since m is odd, $m - 1$ is even. By Lemma 3.1.2 the $m - 1$ roots of $T_{m-1}(x) - x = 0$ are

$$\left\{\cos\left(\frac{2k\pi}{m-2}\right) : 0 \leq k \leq \frac{m-3}{2}\right\} \bigcup \left\{\cos\left(\frac{2k\pi}{m}\right) : 1 \leq k \leq \frac{m-1}{2}\right\}.$$

We partition this set into the following disjoint sets based on the denominators after reducing fractions.

- The $k = 0$ case yields $\cos(0) = 1$. Hence $x - 1$ divides $T_{m-1}(x) - x$.
- For all $3 \leq m' \leq m - 2$ such that m' divides $m - 2$ we have:

$$A_{m',1} = \left\{\cos\left(\frac{2k\pi}{m'}\right) : \left(1 \leq k \leq \frac{m'-1}{2}\right) \wedge \gcd(k,m') = 1\right\}$$

- For all $3 \leq m' \leq m - 1$ such that m' divides m we have:

$$A_{m',1} = \left\{\cos\left(\frac{2k\pi}{m'}\right) : \left(1 \leq k \leq \frac{m'-1}{2}\right) \wedge \gcd(k,m') = 1\right\}$$

- The remaining roots which we want as the roots of $p_m(x)$:

$$A_{m,1} = \left\{ \cos\left(\frac{2k\pi}{m'}\right) : \left(1 \le k \le \frac{m-1}{2}\right) \wedge \gcd(k,m) = 1 \right\}$$

  Since m is odd and our m' in $A_{m',1}$ divides either $m-2$ or m we know m' is odd. Hence by our induction hypothesis there is a polynomial $p_{m'}(x) \in \mathsf{Z}[x]$ of degree $f_{\text{odd}}(m')$ whose roots are the elements of $A_{m',1}$. Hence $p_{m'}$ divides $T_{m-1}(x) - x$.

We take the corresponding polynomials for each $A_{m',1}$.

$$\{p_{m'}(x)\}_{(3\le m'\le m-2)\wedge(m'|m-2)} \bigcup \{p_{m'}(x)\}_{(3\le m'\le m-1)\wedge(m'|m)}$$

Note these polynomials have disjoint sets of roots and all divide $T_{m-1}(x) - x$. By removing these from $T_{m-1}(x) - x$ we have that the following polynomial has exactly $A_{m,1}$ for its roots.

$$p_m(x) = \frac{T_{m-1}(x) - x}{(x-1)(\prod_{3\le m'\le m-2, m'|m-2} p_{m'}(x))(\prod_{3\le m'\le m-1, m'|m} p_{m'}(x))}$$

3) Let k be such that $1 \le k \le \frac{m-1}{2}$ and $\gcd(k,m) = 1$. Note that

$$\cos\left(\frac{(m-2k)\pi}{m}\right) = \cos\left(\frac{-(m-2k)\pi}{m}\right) = -\cos\left(\pi - \frac{-(m-2k)\pi}{m}\right) = -\cos\left(\frac{2k\pi}{m}\right)$$

Hence $A_{m,2} = -A_{m,1}$. Therefore we can take $q_m(x) = p_m(-x)$. ∎

Open Problem 4.3 Let m, k be as in Theorem 4.2.

1. Show that $\cos(2k\pi/m)$ is algebraic of degree exactly $f_{\text{odd}}(m)$.
2. Show that $\cos((m-2k)\pi/m)$ is algebraic of degree exactly $f_{\text{odd}}(m)$.

We will need the next corollary later.

Corollary 4.4 *Let $m \in \mathsf{N}$, $m \ge 3$, be odd. There exists a polynomial $r_m(x) \in \mathsf{Z}[x]$ of degree $m-1$ whose roots are*

$$\left\{ \cos\left(\frac{k\pi}{m}\right) : \left(1 \le k \le m-1\right) \right\}.$$

Proof:

$$r_m(x) = \prod_{m'\ge 2, m'|m} p_{m'}(x). \prod_{m'\ge 3, m'|m, 2\nmid m} q_{m'}(x)$$

∎

5 A Polynomial for $\cos(\frac{k\pi}{m})$: m Even

Theorem 5.1 *Let $m \in \mathsf{N}$, $m \geq 2$, m even.*

1. *There exists $p_m(x) \in \mathsf{Z}[x]$ of degree $f_{\text{even}}(m)$ whose roots are*

$$B_m = \left\{\cos\left(\frac{k\pi}{m}\right) : \left(1 \leq k \leq m\right) \wedge \gcd(k, m) = 1\right\}$$

2. *All of the elements of B_m are algebraic of degree $\leq f_{\text{even}}(m)$. (This follows from Part 1.)*

Proof: To construct $p_m(x)$, m even, we will the same strategy as the previous proof except with a different Chebyshev polynomial. We start with a fraction with $T_{2m-1}(x) - x$ as the numerator. Note that the values of B_m are roots of the polynomial. To remove the other roots we will place previously constructed polynomials $p_{m'}(x)$ $m' < m$ in the denominator.

1) We prove this by induction on m.

Base Case: $m = 2$. Then $B_m = \{\cos(\pi/2)\} = \{0\}$. Let $p_2(x) = x$. $p_2(x)$ has degree 1.

Inductive Hypothesis The theorem is true for all even m', $2 \leq m' < m$.

Inductive Step Since m is even, $2m - 1$ is odd. By Lemma 3.1.2 the $2m - 1$ roots of $T_{2m-1}(x) - x = 0$ are

$$\left\{\cos\left(\frac{2k\pi}{2m-2}\right) : 0 \leq k \leq \frac{2m-2}{2}\right\} \bigcup \left\{\cos\left(\frac{2k\pi}{2m}\right) : 1 \leq k \leq \frac{2m-2}{2}\right\}$$

which is

$$\left\{\cos\left(\frac{k\pi}{m-1}\right) : 0 \leq k \leq m-1\right\} \bigcup \left\{\cos\left(\frac{k\pi}{m}\right) : 1 \leq k \leq m-1\right\}$$

Since m is even, $m - 1$ is odd. Therefore, by Corollary 4.4, there exists $r_{m-1}(x) \in \mathsf{Z}[x]$ whose roots are the elements of the first union.

We partition the second union into the following disjoint sets based on the denominators after reducing fractions.

- The $k = 0$ case. This is just $\cos(0) = 1$. Hence $x - 1$ divides $T_{2m-1}(x) - x$.
- The $k = m - 1$ case. This is just $\cos(\pi) = -1$. Hence $x + 1$ divides $T_{2m-1}(x) - x$.

- For all $2 \le m' \le m-1$ such that m' divides m we have:

$$C_{m'} = \left\{ \cos\left(\frac{k\pi}{m'}\right) : \left(1 \le k \le m'-1\right) \wedge \gcd(k, m') = 1 \right\}.$$

  Here we have two subcases:

  1. If m' is odd then, by Theorem 4.2.3, there exists $p_{m'}(x) * q_{m'}(x) \in \mathsf{Z}[x]$ whose roots are $C_{m'}$. Clearly $p_{m'}(x) * q_{m'}(x)$ divides $T_{2m-1}(x) - x$.
  2. If m' is even then, by the induction hypothesis, there exists $p_{m'}(x) \in \mathsf{Z}[x]$ whose roots are $C_{m'}$. Clearly $p_{m'}(x)$ divides $T_{2m-1}(x) - x$.

- The remaining roots which we want as the roots of our polynomial $p_m(x)$:

$$B_m = \left\{ \cos\left(\frac{k\pi}{m}\right) : \left(1 \le k \le m\right) \wedge \gcd(k, m) = 1 \right\}$$

Using the polynomials described above we attain the following equation with leaves us with the desired roots for pour $p_m(x)$

$$p_m(x) = \frac{T_{2m-1}(x) - x}{(x-1)(x+1)r_{m-1}(x)\prod_{2 \le m' \le m-1, m'|m} p'_m(x) \prod_{2 \le m' \le m-1, m'|m, m \equiv 1 mod 2} q'_m(x)}$$

Clearly the degree of this polynomial is $f_{\text{even}}(m)$. ▮

Open Problem 5.2 Let m, k be as in Theorem 5.1. Show that $\cos(k\pi/m)$ is algebraic of degree exactly $f_{\text{even}}(m)$.

6 What about Sin?

The following is an anonymous post on math stack exchange.

Theorem 6.1 $\sin(2\theta)$ *can not be written as a polynomial over* R *in* $\sin(\theta)$.

Proof: Assume, by way of contradiction, that there exists a polynomial $p(x) \in \mathsf{R}[x]$ such that $\sin(2\theta) = p(\sin(\theta))$. Since $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$ we have

$$2\cos(\theta)\sin(\theta) = \sin(2\theta) = p(\sin(\theta)).$$

Note that if $\theta = 0$ then the left hand side is 0, so $p(\sin(0)) = 0$. Hence $p(0) = 0$. Therefore there exists $q(x) \in \mathsf{R}[x]$ such that $p(x) = xq(x)$. So

$$2\cos(\theta)\sin(\theta) = p(\sin(\theta)) = \sin(\theta)q(\sin(\theta)).$$

$$2\cos(\theta) = p(\sin(\theta)) = q(\sin(\theta)).$$

(We divided by $\sin(\theta)$ so we needed to have $\theta \notin \{n\pi \colon n \in \mathsf{Z}\}$; however, by continuity the two expressions are equal for all θ.)

Square both sides and use $\cos^2(\theta) = 1 - \sin^2(\theta)$ to get

$$4(1 - \sin^2(\theta)) = q(\sin(\theta))^2.$$

The two polynomials $4(1 - x^2)$ and $q(x)^2$ agree for infinitely many x, namely $\sin(\theta)$ as $\theta \in [0, \pi)$. Hence they are equal. But $q(x)^2$ is a square of a polynomial, and $4(1 - x^2) = 4(1 - x)(1 + x)$ is not. Contradiction. ∎

A The First 53 Chebyshev Polynomials

1. $T_1 = x$
2. $T_2 = 2x^2 - 1$
3. $T_3 = 4x^3 - 3x$
4. $T_4 = 8x^4 - 8x^2 + 1$
5. $T_5 = 16x^5 - 20x^3 + 5x$
6. $T_6 = 32x^6 - 48x^4 + 18x^2 - 1$
7. $T_7 = 64x^7 - 112x^5 + 56x^3 - 7x$
8. $T_8 = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$
9. $T_9 = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$
10. $T_{10} = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$
11. $T_{11} = 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x$
12. $T_{12} = 2048x^{12} - 6144x^{10} + 6912x^8 - 3584x^6 + 840x^4 - 72x^2 + 1$
13. $T_{13} = 4096x^{13} - 13312x^{11} + 16640x^9 - 9984x^7 + 2912x^5 - 364x^3 + 13x$
14. $T_{14} = 8192x^{14} - 28672x^{12} + 39424x^{10} - 26880x^8 + 9408x^6 - 1568x^4 + 98x^2 - 1$
15. $T_{15} = 16384x^{15} - 61440x^{13} + 92160x^{11} - 70400x^9 + 28800x^7 - 6048x^5 + 560x^3 - 15x$
16. $T_{16} = 32768x^{16} - 131072x^{14} + 212992x^{12} - 180224x^{10} + 84480x^8 - 21504x^6 + 2688x^4 - 128x^2 + 1$

17. $T_{17} = 65536x^{17} - 278528x^{15} + 487424x^{13} - 452608x^{11} + 239360x^9 - 71808x^7 + 11424x^5 - 816x^3 + 17x$

18. $T_{18} = 131072x^{18} - 589824x^{16} + 1105920x^{14} - 1118208x^{12} + 658944x^{10} - 228096x^8 + 44352x^6 - 4320x^4 + 162x^2 - 1$

19. $T_{19} = 262144x^{19} - 1245184x^{17} + 2490368x^{15} - 2723840x^{13} + 1770496x^{11} - 695552x^9 + 160512x^7 - 20064x^5 + 1140x^3 - 19x$

20. $T_{20} = 524288x^{20} - 2621440x^{18} + 5570560x^{16} - 6553600x^{14} + 4659200x^{12} - 2050048x^{10} + 549120x^8 - 84480x^6 + 6600x^4 - 200x^2 + 1$

21. $T_{21} = 1048576x^{21} - 5505024x^{19} + 12386304x^{17} - 15597568x^{15} + 12042240x^{13} - 5870592x^{11} + 1793792x^9 - 329472x^7 + 33264x^5 - 1540x^3 + 21x$

22. $T_{22} = 2097152x^{22} - 11534336x^{20} + 27394048x^{18} - 36765696x^{16} + 30638080x^{14} - 16400384x^{12} + 5637632x^{10} - 1208064x^8 + 151008x^6 - 9680x^4 + 242x^2 - 1$

23. $T_{23} = 4194304x^{23} - 24117248x^{21} + 60293120x^{19} - 85917696x^{17} + 76873728x^{15} - 44843008x^{13} + 17145856x^{11} - 4209920x^9 + 631488x^7 - 52624x^5 + 2024x^3 - 23x$

24. $T_{24} = 8388608x^{24} - 50331648x^{22} + 132120576x^{20} - 199229440x^{18} + 190513152x^{16} - 120324096x^{14} + 50692096x^{12} - 14057472x^{10} + 2471040x^8 - 256256x^6 + 13728x^4 - 288x^2 + 1$

25. $T_{25} = 16777216x^{25} - 104857600x^{23} + 288358400x^{21} - 458752000x^{19} + 466944000x^{17} - 317521920x^{15} + 146227200x^{13} - 45260800x^{11} + 9152000x^9 - 1144000x^7 + 80080x^5 - 2600x^3 + 25x$

26. $T_{26} = 33554432x^{26} - 218103808x^{24} + 627048448x^{22} - 1049624576x^{20} + 1133117440x^{18} - 825556992x^{16} + 412778496x^{14} - 141213696x^{12} + 32361472x^{10} - 4759040x^8 + 416416x^6 - 18928x^4 + 338x^2 - 1$

27. $T_{27} = 67108864x^{27} - 452984832x^{25} + 1358954496x^{23} - 2387607552x^{21} + 2724986880x^{19} - 2118057984x^{17} + 1143078912x^{15} - 428654592x^{13} + 109983744x^{11} - 18670080x^9 + 1976832x^7 - 117936x^5 + 3276x^3 - 27x$

28. $T_{28} = 134217728x^{28} - 939524096x^{26} + 2936012800x^{24} - 5402263552x^{22} + 6499598336x^{20} - 5369233408x^{18} + 3111714816x^{16} - 1270087680x^{14} + 361181184x^{12} - 69701632x^{10} + 8712704x^8 - 652288x^6 + 25480x^4 - 392x^2 + 1$

29. $T_{29} = 268435456x^{29} - 1946157056x^{27} + 6325010432x^{25} - 12163481600x^{23} + 15386804224x^{21} - 13463453696x^{19} + 8341487616x^{17} - 3683254272x^{15} + 1151016960x^{13} - 249387008x^{11} + 36095488x^9 - 3281408x^7 + 168896x^5 - 4060x^3 + 29x$

30. $T_{30} = 536870912x^{30} - 4026531840x^{28} + 13589544960x^{26} - 27262976000x^{24} + 36175872000x^{22} - 33426505728x^{20} + 22052208640x^{18} - 10478223360x^{16} + 3572121600x^{14} - 859955200x^{12} + 141892608x^{10} - 15275520x^8 + 990080x^6 - 33600x^4 + 450x^2 - 1$

31. $T_{31} = 1073741824x^{31} - 8321499136x^{29} + 29125246976x^{27} - 60850962432x^{25} + 84515225600x^{23} - 82239815680x^{21} + 57567870976x^{19} - 29297934336x^{17} + 10827497472x^{15} - 2870927360x^{13} + 533172224x^{11} - 66646528x^{9} + 5261568x^{7} - 236096x^{5} + 4960x^{3} - 31x$

32. $T_{32} = 2147483648x^{32} - 17179869184x^{30} + 62277025792x^{28} - 135291469824x^{26} + 196293427200x^{24} - 200655503360x^{22} + 148562247680x^{20} - 80648077312x^{18} + 32133218304x^{16} - 9313976320x^{14} + 1926299648x^{12} - 275185664x^{10} + 25798656x^{8} - 1462272x^{6} + 43520x^{4} - 512x^{2} + 1$

33. $T_{33} = 4294967296x^{33} - 35433480192x^{31} + 132875550720x^{29} - 299708186624x^{27} + 453437816832x^{25} - 485826232320x^{23} + 379364311040x^{21} - 218864025600x^{19} + 93564370944x^{17} - 29455450112x^{15} + 6723526656x^{13} - 1083543552x^{11} + 118243840x^{9} - 8186112x^{7} + 323136x^{5} - 5984x^{3} + 33x$

34. $T_{34} = 8589934592x^{34} - 73014444032x^{32} + 282930970624x^{30} - 661693399040x^{28} + 1042167103488x^{26} - 1167945891840x^{24} + 959384125440x^{22} - 586290298880x^{20} + 267776819200x^{18} - 91044118528x^{16} + 22761029632x^{14} - 4093386752x^{12} + 511673344x^{10} - 42170880x^{8} + 2108544x^{6} - 55488x^{4} + 578x^{2} - 1$

35. $T_{35} = 17179869184x^{35} - 150323855360x^{33} + 601295421440x^{31} - 1456262348800x^{29} + 2384042393600x^{27} - 2789329600512x^{25} + 2404594483200x^{23} - 1551944908800x^{21} + 754417664000x^{19} - 275652608000x^{17} + 74977509376x^{15} - 14910300160x^{13} + 2106890240x^{11} - 202585600x^{9} + 12403200x^{7} - 434112x^{5} + 7140x^{3} - 35x$

36. $T_{36} = 34359738368x^{36} - 309237645312x^{34} + 1275605286912x^{32} - 3195455668224x^{30} + 5429778186240x^{28} - 6620826304512x^{26} + 5977134858240x^{24} - 4063273943040x^{22} + 2095125626880x^{20} - 819082035200x^{18} + 240999137280x^{16} - 52581629952x^{14} + 8307167232x^{12} - 916844544x^{10} + 66977280x^{8} - 2976768x^{6} + 69768x^{4} - 648x^{2} + 1$

37. $T_{37} = 68719476736x^{37} - 635655159808x^{35} + 2701534429184x^{33} - 6992206757888x^{31} + 12315818721280x^{29} - 15625695002624x^{27} + 14743599316992x^{25} - 10531142369280x^{23} + 5742196162560x^{21} - 2392581734400x^{19} + 757650882560x^{17} - 180140769280x^{15} + 31524634624x^{13} - 3940579328x^{11} + 336540160x^{9} - 18356736x^{7} + 573648x^{5} - 8436x^{3} + 37x$

38. $T_{38} = 137438953472x^{38} - 1305670057984x^{36} + 5712306503680x^{34} - 15260018802688x^{32} + 27827093110784x^{30} - 36681168191488x^{28} + 36108024938496x^{26} - 27039419596800x^{24} + 15547666268160x^{22} - 6880289095680x^{20} + 2334383800320x^{18} - 601280675840x^{16} + 115630899200x^{14} - 16188325888x^{12} + 1589924864x^{10} - 103690752x^{8} + 4124064x^{6} - 86640x^{4} + 722x^{2} - 1$

39. $T_{39} = 274877906944x^{39} - 2680059592704x^{37} + 12060268167168x^{35} - 33221572034560x^{33} + 62646392979456x^{31} - 85678155104256x^{29} + 87841744879616x^{27} - 68822438510592x^{25} + 41626474905600x^{23} - 19502774353920x^{21} + 7061349335040x^{19} - 1960212234240x^{17} + 411402567680x^{15} - 63901286400x^{13} + 7120429056x^{11} - 543921664x^{9} + 26604864x^{7} - 746928x^{5} + 9880x^{3} - 39x$

40. $T_{40} = 549755813888x^{40} - 5497558138880x^{38} + 25426206392320x^{36} - 72155450572800x^{34} + 140552804761600x^{32} - 199183403319296x^{30} + 212364657950720x^{28} - 173752901959680x^{26} +$

$110292369408000x^{24} - 54553214976000x^{22} + 21002987765760x^{20} - 6254808268800x^{18} + 1424085811200x^{16} - 243433472000x^{14} + 30429184000x^{12} - 2677768192x^{10} + 156900480x^{8} - 5617920x^{6} + 106400x^{4} - 800x^{2} + 1$

41. $T_{41} = 1099511627776x^{41} - 11269994184704x^{39} + 53532472377344x^{37} - 156371169312768x^{35} + 314327181557760x^{33} - 461013199618048x^{31} + 510407471005696x^{29} - 435347548798976x^{27} + 289407177326592x^{25} - 150732904857600x^{23} + 61508749885440x^{21} - 19570965872640x^{19} + 4808383856640x^{17} - 898269511680x^{15} + 124759654400x^{13} - 12475965440x^{11} + 857722624x^{9} - 37840704x^{7} + 959728x^{5} - 11480x^{3} + 41x$

42. $T_{42} = 2199023255552x^{42} - 23089744183296x^{40} + 112562502893568x^{38} - 338168545017856x^{36} + 700809813688320x^{34} - 1062579203997696x^{32} + 1219998345330688x^{30} - 1083059755548672x^{28} + 752567256612864x^{26} - 411758179123200x^{24} + 177570714746880x^{22} - 60144919511040x^{20} + 15871575982080x^{18} - 3220624834560x^{16} + 492952780800x^{14} - 55381114880x^{12} + 4393213440x^{10} - 232581888x^{8} + 7537376x^{6} - 129360x^{4} + 882x^{2} - 1$

43. $T_{43} = 4398046511104x^{43} - 47278999994368x^{41} + 236394999971840x^{39} - 729869562413056x^{37} + 1557990796689408x^{35} - 2439485589553152x^{33} + 2901009890279424x^{31} - 2676526982103040x^{29} + 1940482062024704x^{27} - 1112923535572992x^{25} + 505874334351360x^{23} - 181798588907520x^{21} + 51314117836800x^{19} - 11249633525760x^{17} + 1884175073280x^{15} - 235521884160x^{13} + 21262392320x^{11} - 1322886400x^{9} + 52915456x^{7} - 1218448x^{5} + 13244x^{3} - 43x$

44. $T_{44} = 8796093022208x^{44} - 96757023244288x^{42} + 495879744126976x^{40} - 1572301627719680x^{38} + 3454150138396672x^{36} - 5579780992794624x^{34} + 6864598984556544x^{32} - 6573052309536768x^{30} + 4964023879598080x^{28} - 2978414327758848x^{26} + 1423506847825920x^{24} - 541167892561920x^{22} + 162773155184640x^{20} - 38370843033600x^{18} + 6988974981120x^{16} - 963996549120x^{14} + 97905899520x^{12} - 7038986240x^{10} + 338412800x^{8} - 9974272x^{6} + 155848x^{4} - 968x^{2} + 1$

45. $T_{45} = 17592186044416x^{45} - 197912092999680x^{43} + 1039038488248320x^{41} - 3380998255411200x^{39} + 7638169839206400x^{37} - 12717552782278656x^{35} + 16168683558666240x^{33} - 16047114509352960x^{31} + 12604574741299200x^{29} - 7897310717542400x^{27} + 3959937231224832x^{25} - 1588210119475200x^{23} + 507344899276800x^{21} - 128055803904000x^{19} + 25227583488000x^{17} - 3812168171520x^{15} + 431333683200x^{13} - 35340364800x^{11} + 1999712000x^{9} - 72864000x^{7} + 1530144x^{5} - 15180x^{3} + 45x$

46. $T_{46} = 35184372088832x^{46} - 404620279021568x^{44} + 2174833999740928x^{42} - 7257876254949376x^{40} + 16848641306132480x^{38} - 28889255702953984x^{36} + 37917148110127104x^{34} - 38958828003262464x^{32} + 31782201792135168x^{30} - 20758645314682880x^{28} + 10898288790208512x^{26} - 4599927086776320x^{24} + 1555857691115520x^{22} - 418884762992640x^{20} + 88826010009600x^{18} - 14613311324160x^{16} + 1826663915520x^{14} - 168586629120x^{12} + 11038410240x^{10} - 484140800x^{8} + 13034560x^{6} - 186208x^{4} + 1058x^{2} - 1$

47. $T_{47} = 70368744177664x^{47} - 826832744087552x^{45} + 4547580092481536x^{43} - 15554790998147072x^{41} + 37078280867676160x^{39} - 65416681245114368x^{37} + 88551849002532864x^{35} - 94086339565191168x^{33} + 79611518093623296x^{31} - 54121865370664960x^{29} + 29693888297959424x^{27} - 13159791404777472x^{25} +$

$4699925501706240x^{23}-1345114425262080x^{21}+305707823923200x^{19}-54454206136320x^{17}+7465496002560x^{15}-768506941440x^{13}+57417185280x^{11}-2967993600x^{9}+98933120x^{7}-1902560x^{5}+17296x^{3}-47x$

48. $T_{48} = 140737488355328x^{48}-1688849860263936x^{46}+9499780463984640x^{44}-33284415996035072x^{42}+81414437990301696x^{40}-147682003796361216x^{38}+205992953708019712x^{36}-226089827240509440x^{34}+198181864190509056x^{32}-140025932533465088x^{30}+80146421910601728x^{28}-37217871599763456x^{26}+13999778090188800x^{24}-4246086541639680x^{22}+1030300410839040x^{20}-197734422282240x^{18}+29544303329280x^{16}-3363677798400x^{14}+283420999680x^{12}-16974397440x^{10}+682007040x^{8}-16839680x^{6}+220800x^{4}-1152x^{2}+1$

49. $T_{49} = 281474976710656x^{49}-3448068464705536x^{47}+19826393672056832x^{45}-71116412084551680x^{43}+178383666978750464x^{41}-332442288460398592x^{39}+477402588661153792x^{37}-540731503483551744x^{35}$
$490450067946209280x^{33}-359663383160553472x^{31}+214414709191868416x^{29}-104129631497486336x^{27}$
$41159347585155072x^{25}-13192098584985600x^{23}+3405715246940160x^{21}-701176668487680x^{19}+113542812794880x^{17}-14192851599360x^{15}+1335348940800x^{13}-91365980160x^{11}+4332007680x^{9}-132612480x^{7}+2344160x^{5}-19600x^{3}+49x$

50. $T_{50} = 562949953421312x^{50}-7036874417766400x^{48}+41341637204377600x^{46}-151732604633088000x^{44}$
$390051749953536000x^{42}-746299014911098880x^{40}+1102487181118668800x^{38}-1287455960675123200x$
$1206989963132928000x^{34}-917508630511616000x^{32}+568855350917201920x^{30}-288405684905574400x^{2}$
$119536566770073600x^{26}-40383975260160000x^{24}+11057517035520000x^{22}-2432653747814400x^{20}+424820047872000x^{18}-57930006528000x^{16}+6034375680000x^{14}-466152960000x^{12}+25638412800x^{10}-947232000x^{8}+21528000x^{6}-260000x^{4}+1250x^{2}-1$

51. $T_{51} = 1125899906842624x^{51}-14355223812243456x^{49}+86131342873460736x^{47}-323291602938232832x$
$851219911991623680x^{43}-1670981696800948224x^{41}+2537416650697736192x^{39}-3052314510011400192$
$2954711429749407744x^{35}-2325467328969441280x^{33}+1497374084994957312x^{31}-791226079003017216$
$343202765037633536x^{27}-121927298105475072x^{25}+35307132656025600x^{23}-8271022742568960x^{21}+1550816764231680x^{19}-229402825850880x^{17}+26261602959360x^{15}-2267654860800x^{13}+142642805760x^{11}-6226471680x^{9}+175668480x^{7}-2864160x^{5}+22100x^{3}-51x$

52. $T_{52} = 2251799813685248x^{52}-29273397577908224x^{50}+179299560164687872x^{48}-687924843080843264$
$1854172428616335360x^{44}-3732015143555432448x^{42}+5821132316306571264x^{40}-720711620114146918$
$7196878820173938688x^{36}-5857924621071810560x^{34}+3912256800501530624x^{32}-215130750892323635$
$974811214980841472x^{28}-363391162981023744x^{26}+110998240572211200x^{24}-27599562520657920x^{22}+5534287276277760x^{20}-883625699573760x^{18}+110453212446720x^{16}-10569685401600x^{14}+751438571520x^{12}-38091356160x^{10}+1298568960x^{8}-27256320x^{6}+304200x^{4}-1352x^{2}+1$

53. $T_{53} = 4503599627370496x^{53}-59672695062659072x^{51}+372954344141619200x^{49}-146198102903514726$
$403163646017090355 2x^{45}-8315250199102488576x^{43}+13313246329414090752x^{41}-16951649052980674$
$17446072150359277568x^{37}-14670560671893028864x^{35}+10149980929972502528x^{33}-5799989102841430016x^{31}+2740848508964700160x^{29}-1069985090999681024x^{27}+343923779249897472$
$90506257697341440x^{23}+19339597295124480x^{21}-3318068163379200x^{19}+450309250744320x^{17}-$

$47400973762560x^{15}+3770532003840x^{13}-218825518080x^{11}+8823609600x^{9}-230181120x^{7}+3472560x^{5}-24804x^{3}+53x$

B Table of Polynomials

In the first column if we have a number line $\pi/4$ we mean $\cos(\pi/4)$.

Roots	Poly
$\pi/2$	$x-1$
$\pi/3$	$-2x+1$
$2\pi/3$	$2x+1$
$\pi/4, 3\pi/4$	$2x^2-1$
$\pi/5, 3\pi/5$	$4x^2-2x-1$
$2\pi/5, 4\pi/5$	$4x^2+2x-1$
$\pi/6, 5\pi/6$	$4x^2-3$
$\pi/7, 3\pi/7, 5\pi/7$	$-8x^3+4x^2+4x-1$
$2\pi/7, 4\pi/7, 6\pi/7$	$8x^3+4x^2-4x-1$
$\pi/8, 3\pi/8, 5\pi/8, 7\pi/8$	$8x^4-8x^2+1$
$\pi/9, 5\pi/9, 7\pi/9,$	$-8x^3+6x+1$
$2\pi/9, 4\pi/9, 8\pi/9$	$8x^3-6x+1$
$\pi/10, 3\pi/10, 7\pi/10, 9\pi/10$	$16x^4-20x^2+5$
$\pi/11, 3\pi/11, 5\pi/11, 7\pi/11, 9\pi/11$	$-32x^5+16x^4+32x^3-12x^2-6x+1$
$2\pi/11, 4\pi/11, 6\pi/11, 8\pi/11, 10\pi/11$	$32x^5+16x^4-32x^3-12x^2+6x+1$
$\pi/12, 5\pi/12, 7\pi/12, 11\pi/12$	$16x^4-16x^2+1$
$\pi/13, 3\pi/13, 5\pi/13, 7\pi/13, 9\pi/13, 11\pi/13$	$64x^6-32x^5-80x^4+32x^3+24x^2-6x-1$
$2\pi/13, 4\pi/13, 6\pi/13, 8\pi/13, 10\pi/13, 12\pi/13$	$64x^6+32x^5-80x^4-32x^3+24x^2+6x-1$
$\pi/14, 3\pi/14, 5\pi/14, 9\pi/14, 11\pi/14, 13\pi/14$	$64x^6+112x^4-56x^2+7$
$\pi/15, 7\pi/15, 11\pi/15, 13\pi/15$	$16x^4+8x^3-16x^2-8x+1$
$2\pi/15, 4\pi/15, 8\pi/15, 14\pi/15$	$16x^4-8x^3-16x^2+8x+1$
$\pi/16, 3\pi/16, 5\pi/16, 7\pi/16, 9\pi/16\ldots$	$128x^8-256x^6+160x^4-32x^2+1$
$\pi/17, 3\pi/17, 5\pi/17, \ldots$	$256x^8-128x^7-448x^6+192x^5$ $+240x^4-80x^3-40x^2+8x+1$
$2\pi/17, 4\pi/17, 6\pi/17, 14\pi\ldots$	$256x^8+128x^7-448x^6-192x^5$ $+240x^4+80x^3-40x^2-8x+1$
$\pi/18, 3\pi/18, 5\pi/18, 7\pi/18, 9\pi/18\ldots$	$-64x^6+96x^4-36x^2+3$
$\pi/19, 3\pi/19, 5\pi/19, \ldots$	$-512x^9+256x^8+1024x^7-448x^6-672x^5$ $+240x^4+160x^3-40x^2-10x+1$
$2\pi/19, 4\pi/19, 6\pi/19, 8\pi/19\ldots$	$512x^9+256x^8-1024x^7-448x^6+672x^5$ $+240x^4-160x^3-40x^2+10x+1$
$\pi/20, 3\pi/20, 5\pi/20, 7\pi/20, 9\pi/20\ldots$	$256x^8-512x^6+304x^4-48x^2+1$
$\pi/21, 5\pi/21, 7\pi/21, \ldots$	$64x^6+32x^5-96x^4-48x^3+32x^2+16x+1$
$2\pi/21, 4\pi/21, 6\pi/21, 8\pi/21\ldots$	$64x^6-32x^5-96x^4+48x^3+32x^2-16x+1$

Roots	Poly
$\pi/22, 3\pi/22, 5\pi/22, 7\pi/22, 9\pi/22 \ldots$	$1024x^{10} - 2816x^8 + 2816x^6 - 1232x^4 + 220x^2 + 11$
$\pi/23, 3\pi/23, 5\pi/23, \ldots$	$-2048x^{11} + 1024x^{10} + 5120x^9 - 2304x^8 - 4608x^7 + 1792x^6$ $+1792x^5 - 560x^4 - 280x^3 + 60x^2 + 12x - 1$
$2\pi/23, 4\pi/23, 6\pi/23, 8\pi/23 \ldots$	$2048x^{11} + 1024x^{10} - 5120x^9 - 2304x^8 + 4608x^7 + 1792x^6$ $-1792x^5 - 560x^4 + 280x^3 + 60x^2 - 12x - 1$
$\pi/24, 3\pi/24, 5\pi/24, 7\pi/24, 9\pi/24 \ldots$	$256x^8 - 512x^6 + 320x^4 - 64x^2 + 1$
$\pi/25, 3\pi/25, 7\pi/25, \ldots$	$1024x^{10} - 2560x^8 + 2240x^6 - 32x^5$ $-800x^4 + 40x^3 + 100x^2 - 10x - 1$
$2\pi/25, 4\pi/25, 6\pi/25, 8\pi/25 \ldots$	$1024x^{10} - 2560x^8 + 2240x^6 + 32x^5$ $-800x^4 - 40x^3 + 100x^2 + 10x - 1$
$\pi/26, 3\pi/26, 5\pi/26, 7\pi/26, 9\pi/26 \ldots$	$4096x^{12} - 13312x^{10} + 16640x^8 - 9984x^6$ $+2912x^4 - 364x^2 + 13$
$\pi/27, 7\pi/27, 11\pi/27 \ldots$	$-512x^9 + 1152x^7 - 864x^5 + 240x^3 - 18x + 1$
$2\pi/27, 4\pi/27, 6\pi/27, 8\pi/27 \ldots$	$512x^9 - 1152x^7 + 864x^5 - 240x^3 + 18x + 1$

References

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