Let $\zeta_n = e^{2\pi i/n}$ be a primitive $n$th root of unity and let $\gcd(k, n) = 1$. Then $\zeta_n^k$ is also a primitive $n$th root of unity. Therefore, the field $\mathbb{Q}(\zeta_n^k)$ has degree $\phi(n)$ over $\mathbb{Q}$. Complex conjugation is an automorphism of order 2 and

$$\cos(2\pi k/n) = \frac{1}{2}(\zeta_n^k + \zeta_n^{-k})$$

is fixed by complex conjugation. Therefore, $\mathbb{Q}(\zeta_n^k)$ is degree greater than one over the field $\mathbb{Q}(\cos(2\pi k/n))$. Since $\zeta_n^k$ is a root of the quadratic equation

$$X^2 - 2\cos(2\pi k/n)X + 1 = 0,$$

the field $\mathbb{Q}(\zeta_n^k)$ is degree at most 2 over the field $\mathbb{Q}(\cos(2\pi k/n))$, hence of degree exactly 2. Therefore, $\cos(2\pi k/n)$ is algebraic of degree exactly $\phi(n)/2$ over $\mathbb{Q}$.

Now consider $\cos(\pi a/b)$, where $\gcd(a, b) = 1$. If $a$ is odd, rewrite this as $\cos(2\pi a/2b)$, which has degree $\frac{1}{2}\phi(2b)$ over $\mathbb{Q}$, since $\gcd(a, 2b) = 1$. If $a$ is even, let $a = 2a'$ and rewrite as $\cos(2\pi a'/b)$. Since $\gcd(a', b)$ divides $\gcd(a, b) = 1$, we have $\gcd(a', b) = 1$, so the cosine has degree $\frac{1}{2}\phi(b)$ over $\mathbb{Q}$.

So the final answer is: $\cos(\pi a/b)$ is algebraic of degree $\frac{1}{2}\phi(2b)$ is $a$ is odd and is algebraic of degree $\frac{1}{2}\phi(b)$ when $b$ is even.

Note that if $b$ is odd, then $\phi(2b) = \phi(b)$, so the degree is $\frac{1}{2}\phi(b)$ for both parities of $a$.

The case of sine is similar, but more complicated. Given $n$, let $n' = \operatorname{lcm}(4, n)$ and let $\zeta = e^{2\pi i/n'}$ be a primitive $n'$th root of unity. Let $\gcd(k, n') = 1$, so $\zeta^k$ is also a primitive $n'$th root of unity. Then $\sin(2\pi k/n')$ is in $\mathbb{Q}(\zeta)$ and is fixed by complex conjugation, so $\mathbb{Q}(\zeta)$ has degree at least 2 over $\mathbb{Q}(\sin(2\pi k/n'))$. But we have the sequence of fields

$$\mathbb{Q}(\sin(2\pi k/n')) \subset \mathbb{Q}(\sin(2\pi k/n'), i) \subseteq \mathbb{Q}(\zeta).$$

Since $i$ is a root of $X^2 + 1 = 0$, the first containment is of degree exactly 2. Since $\zeta^k$ is a root of the quadratic equation $X^2 - 2i\sin(2\pi k/n')X + 1$, the second inclusion is of degree 1 or 2.

I believe that the latter degree is 1 when $n'$ is a power of 2 and is 2 otherwise, but I haven't checked this. In any case, we get that $\sin(2\pi k/n')$ is algebraic of degree either $\frac{1}{2}\phi(n')$ or $\frac{1}{4}\phi(n')$. This can be converted to a statement about $\sin(\pi a/b)$.