The Algebraic Degree of $\sin(\frac{a\pi}{m})$
An Exposition by
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1 Sine

We have shown the following:

1. If $a, b \in \mathsf{Z}$, $\gcd(a, b) = 1$, and a is odd, then $\deg(\cos(a\pi/b)) = \phi(2b)/2$
2. If $a, b \in \mathsf{Z}$, $\gcd(a, b) = 1$, and a is even, then $\deg\cos((a\pi/b)) = \phi(b)/2$.

We use this to get formulas for the algebraic degree of sin.
Let $a, b \in \mathsf{Z}$, $\gcd(a, b) = 1$.
$\sin(a\pi/b) = \cos(\pi/2 - a\pi/b) = \cos((b - 2a)\pi/2b)$.
We could stop here and just say

$$\deg(\sin(a\pi/b)) = \deg(\cos((b - 2a)\pi/2b)).$$

However, we will see if we can get a formula out of this.
Note that $b - 2a$ and $2b$ need not be relatively prime. Let $\gcd(b - 2a, 2b) = d$.
Since d divides $b - 2a$ and d divides $2b$, d divides $4a$.

Case 1: b odd. If b is odd then since d divides $2b$, d divides b. Hence d is odd. Since d divides $4a$, d divides a. Hence $d = 1$, so $b - 2a$ and $2b$ are relatively prime. Since b is odd, $b - 2a$ is odd. Hence:

$$\deg(\sin(a\pi/b)) = \deg(\cos((b - 2a)\pi/2b)) = \phi(4b)/2.$$

Case 2: b is even. Then a is odd (else a, b are both even so not relatively prime).

1. If $(b - 2a)/d$ is odd then

$$\deg(\sin(a\pi/b)) = \deg(\cos((b-2a)\pi/2b)) = \deg(\cos(((b-2a)/d)\pi/(2b/d))) = \phi(4b/d)/2.$$

2. If $(b - 2a)/d$ is even then

$$\deg(\sin(a\pi/b)) = \deg(\cos((b-2a)\pi/2b)) = \deg(\cos(((b-2a)/d)\pi/(2b/d))) = \phi(2b/d)/2.$$