Cut and Choose Misc.
Exposition by William Gasarch

1 Introduction

We investigate some aspects of two-person cake cutting. For this exposition the term *protocol* means *2-person cake cutting protocol.*

2 ϵ-Proportional Super Cheat Proof

Def 2.1 A protocol is *super cheat proof* if even if Alice (or Bob) knows Bob (or Alice's) tastes, cheating may lead to a worse outcome for her.

Obtaining this seems very hard. Hence we weaken another property.

Def 2.2 A protocol is *ϵ-proportional* if each player gets within ϵ of $\frac{1}{2}$.

Theorem 2.3 *For all L there is a a $\frac{1}{L}$-proportional super cheat proof protocol. The number of cuts is $O(L^2)$.*

Proof:

1. Alice cuts the cake pie into $2L$ pieces. (Evenly, so each piece is $\frac{1}{2L}$.) (Later I will point out why we use $\frac{1}{2L}$.) This takes $O(L)$ cuts.
2. Bob cuts each piece into $2L$ pieces. (Evenly. If both players played honestly then each piece is $\le \frac{1}{2L}$.)
3. Alice and Bob write down privately and simultaneously what each piece is worth to them. They reveal these numbers.
4. Let the pieces be $p_1, \ldots, p_m$. Let $\mathrm{V}_A(p_i)$ be how much Alice values piece p_i. Let $\mathrm{V}_B(p_i)$ be how much Bob values piece p_i.
5. Alice and Bob reorder the pieces:
    - Let $q_1 = p_1$.
    - Assume that $q_1, \ldots, q_k$ are already defined.

– If

$$\sum_{i=1}^{k} \mathrm{V}_A(q_i) < \sum_{i=1}^{k} \mathrm{V}_B(q_i)$$

then Alice and Bob find a piece p not already used such that $\mathrm{V}_A(p) < \mathrm{V}_B(p)$. Let $q_{k+1} = p$.

– If

$$\sum_{i=1}^{k} \mathrm{V}_B(q_i) < \sum_{i=1}^{k} \mathrm{V}_A(q_i)$$

then Alice and Bob find a piece p not already used such that $\mathrm{V}_B(p) < \mathrm{V}_A(p)$. Let $q_{k+1} = p$.

– If

$$\sum_{i=1}^{k} \mathrm{V}_A(q_i) = \sum_{i=1}^{k} \mathrm{V}_B(q_i)$$

then Alice and Bob find a piece p not already used such that $\mathrm{V}_B(p) \leq \mathrm{V}_A(p)$. Let $q_{k+1} = p$. (They could have done $\mathrm{V}_A(p) \geq \mathrm{V}_B(p)$ it would not have mattered.)

NOTE- it is an exercise to show that they can always find such a piece p.

6. We now have $q_1, \ldots, q_m$. Let k be the least number such that

$$\sum_{i=1}^{k} \mathrm{V}_A(q_i) \geq \frac{1}{2}.$$

7. Let

$$P = q_1 \cup \cdots \cup q_k$$

$$Q = q_{k+1} \cup \cdots \cup a_n$$

8. FLIP A COIN!

If its HEADS then Alice gets P, Bob gets Q.

If its TAILS then Bob gets P Alice gets Q.

We show that if both follow the advice then P and Q are both very close to $\frac{1}{2}$ in both measures.

If someone does not follow the advice, even if they know the other players tastes, they could do worse not knowing who will get P and who will get Q. Hence, once we establish that following the advice leads Alice and Bob both thinking P and Q are very close to $\frac{1}{2}$ in value, we will have that this protocol is super-cheat proof.

Notation 2.4 $S_A^j = \sum_{i=1}^j \mathrm{V}_A(q_i)$. $S_B^j = \sum_{i=1}^j \mathrm{V}_B(q_i)$. Note that $\mathrm{V}_A(P) = S_A^k$ and $\mathrm{V}_B(P) = S_B^k$.

KEY FACT: For all j,

$$|S_A^j - S_B^j| \leq \frac{1}{2L}.$$

Proof of KEY FACT: We first show that this holds at $j = 1$. All this means is that $|\mathrm{V}_A(q_1) - \mathrm{V}_B(q_1)| \leq \frac{1}{2L}$. Since each of these values if between 0 and $\frac{1}{2L}$ the difference is at most $\frac{1}{2L}$

Intuitively, whenever one of the sums is behind we define q so that it catches up. This is why the difference of the sums can never get to large. We show this formally by a technique known as induction. We show that if the statement holds for j then it holds for $j + 1$. This is enough to establish that it holds for all j.

Assume

$$|S_A^j - S_B^j| \leq \frac{1}{2L}.$$

Assume that $S_A^j > S_B^j$, hence $|S_A^j - S_B^j| = S_A^j - S_B^j$. The other cases are similar. Alice and Bob then find a p such that $\mathrm{V}_A(p) < \mathrm{V}_B(p)$ and let $q_{j+1} = p$. Note that

$S_A^{j+1} = S_A^j + \mathrm{V}_A(p)$

$S_B^{j+1} = S_B^j + \mathrm{V}_B(p)$

Also note that

$$0 \leq S_A^j - S_B^j \leq \frac{1}{2L}$$

$$-\frac{1}{2L} \leq \mathrm{V}_A(p) - \mathrm{V}_B(p) \leq 0.$$

Add these 2 together to get

$$-\frac{1}{2L} \le S_A^j + \mathrm{V}_A(p) - S_B^j - \mathrm{V}_B(p) \le \frac{1}{2L}$$

$$-\frac{1}{2L} \le (S_A^j + \mathrm{V}_A(p)) - (S_B^j + \mathrm{V}_B(p)) \le \frac{1}{2L}.$$

$$-\frac{1}{2L} \le S_A^{j+1} - S_B^{j+1} \le \frac{1}{2L}.$$

Hence

$$|S_A^{j+1} - S_B^{j+1}| \le \frac{1}{2L}.$$

End of Proof of KEY FACT

We now show that $\mathrm{V}_A(P)$ is within $\frac{1}{L}$ of $1/2$ (which will not use KEY FACT) and that $\mathrm{V}_B(Q)$ is within $\frac{1}{L}$ of $1/2$ (which will use KEY FACT).

ALICE: By the definition of k, Alice gets $P = S_A^k$ where $S_A^{k-1} < \frac{1}{2}$ and $S_A^k \ge \frac{1}{2}$. Note that

$$S_A^k = S_A^{k-1} + \mathrm{V}_A(q_k)$$

Since $S_A^{k-1} < \frac{1}{2}$ and $\mathrm{V}_A(q_k) \le \frac{1}{2L}$

$$S_A^k = S_A^{k-1} + \mathrm{V}_A(q_k) \le \frac{1}{2} + \frac{1}{2L}.$$

Combine this with $S_A^k \ge \frac{1}{2}$ to obtain

$$\frac{1}{2} \le S_A^k \le \frac{1}{2} + \frac{1}{2L}.$$

Hence

$$\frac{1}{2} \le \mathrm{V}_A(P) \le \frac{1}{2} + \frac{1}{2L}.$$

This is stronger than we need which is just that $\mathrm{V}_A(P)$ is within $\frac{1}{L}$ of $\frac{1}{2}$.

Since Alice things P has value close to $\frac{1}{2}$ she must also think that Q has value close to $\frac{1}{2}$. We omit details.

BOB: We will show that (1) $\mathrm{V}_A(Q)$ is to $\frac{1}{2}$, (2) $\mathrm{V}_A(Q)$ and $\mathrm{V}_B(Q)$ are close hence (3) $\mathrm{V}_B(Q)$ is close to $\frac{1}{2}$.

Recall that

$$\frac{1}{2} \leq \mathrm{V}_A(P) \leq \frac{1}{2} + \frac{1}{2L}.$$

Also recall from KEY FACT that

$$-\frac{1}{2L} \leq S_B^k - S_A^k \leq \frac{1}{2L}$$

so

$$-\frac{1}{2L} \leq \mathrm{V}_B(P) - \mathrm{V}_A(P) \leq \frac{1}{2L}$$

Adding these 2 equations we get

$$\frac{1}{2} - \frac{1}{2L} \leq \mathrm{V}_B(P) \leq \frac{1}{2} + \frac{1}{2L} + \frac{1}{2L} = \frac{1}{2} + \frac{1}{L}$$

(NOTE- this is why we picked $\frac{1}{2L}$ in the first place— since we knew that we would end up within twice that value which is $\frac{1}{L}$.)

We are really interested in $\mathrm{V}_B(Q) = 1 - \mathrm{V}_B(P)$. So lets negate all sides and add 1.

First we negate all sides and switch the inequality:

$$-\frac{1}{2} + \frac{1}{2L} \geq -\mathrm{V}_B(P) \geq -\frac{1}{2} - \frac{1}{L}$$

Now we rewrite the inequality:

$$-\frac{1}{2} - \frac{1}{L} \leq -\mathrm{V}_B(P) \leq -\frac{1}{2} + \frac{1}{2L}$$

Now we add 1 to all sides:

$$1 - \frac{1}{2} - \frac{1}{L} \leq 1 - \mathrm{V}_B(P) \leq 1 - \frac{1}{2} + \frac{1}{2L}$$

$$\frac{1}{2} - \frac{1}{L} \leq \mathrm{V}_B(Q) \leq \frac{1}{2} + \frac{1}{2L}.$$

Since Bob thinks Q has value close to $\frac{1}{2}$ she must also think that P has value close to $\frac{1}{2}$. We omit details.

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3 Is there a Protocol where Both Get $\frac{1}{2}$

The key to the ϵ-proportional super-cheat proof was getting two pieces that both thought were *about* $\frac{1}{2}$. If there is a protocol that gives both *exactly* $\frac{1}{2}$ then we can get proportional super-cheat proof. Alas there is not.

Theorem 3.1 *For all k there is no protocol such that, at the end, both parties have exactly 1/2.*

Proof sketch:

We first show there is no 1-cut protocol. We can assume Alice cuts. We can then assign Bob to like one piece 1/4 and one piece 3/4.

What about a 2-cut 2-person protocol? Assume that Alice makes the first cut. We can set Bob's values so that the scenario is as follows:

	P_1	P_2
Alice	x_1	x_2
Bob	2/3	1/3

We now assume that Bob makes the next cut. If he cuts P_2 then no set of pieces can, in Bob's eyes, add up to 1/2. Hence we assume Bob cuts P_1.

	P_{11}	P_{12}	P_2
Alice	x_{11}	x_{12}	x_2
Bob	y_1	y_2	1/3

We look at all the ways Bob's pieces COULD add up to 1/2 and make sure to set Alice's values so that none of them occur.

1. $y_1 = 1/6$. We need to make $x_{11} + x_2 \neq 1/2$. We can control x_{11} and x_{12} (so long as they add up to x_1) but not x_2. Hence we rewrite this as $x_{11} \neq 1/2 - x_2$.

2. $y_2 = 1/6$. We need to make $x_{12} + x_2 \neq 1/2$. We can control x_{11} and x_{12} (so long as they add up to x_1) but not x_2. Hence we rewrite this as $x_{12} \neq 1/2 - x_2$.

Hence we need only pick P_{11} and P_{12} so that $x_{11} \neq 1/2 - x_2$ and $x_{12} \neq 1/2 - x_2$. There are an infinite (x_{11}, x_{12}) such that $x_{11} + x_{12} = x_2$, and we only need to avoid a few values. Hence this can be done.

This kind of proof can be extended to 3-cuts, 4-cuts, etc.

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4 A Proportional Super Cheat Proof Protocol

We just proved that there was no protocol that produces two exactly equal pieces. We will now give a protocol that produces two exactly equal pieces. Say what?

The proof that no such protocol existed actually showed that no *discrete* protocol exists. We will give a *moving knife* protocol which is really a *continuous* protocol.

Theorem 4.1 *There is a Moving Knife protocol where, at the end, they both have exactly 1/2. Hence there is proportional super cheat proof MK protocol*

Proof:

1. Alice places one knife on the left end of the cake and the other such that the cake between the two is 1/2.
2. Alice moves the two knifes always keeping 1/2 between them.
3. Bob yells STOP when he thinks that the inside and outside are both 1/2.
4. They decide who gets inside vs outside with a coin toss.

If Bob first thought that the inside was $< 1/2$ and he outside was $> 1/2$ then there must be a point where he thinks they are equal (by either the intermediate value theorem or common sense).

Since they do not know which person gets which piece they are both motivated. to follow the advice of the protocol. Note that this holds even if Alice knows Bob's tastes or vice-versa. ∎