**250 FINAL**

**<u>Do not open this exam until you are told.</u> Read these instructions:**

1. This is a closed book exam, though ONE sheet of notes is allowed. **No calculators, or other aids are allowed**. If you have a question during the exam, please raise your hand.
2. There are 5 problems which add up to 100 points. The exam is 2 hours. (You shouldn't need that much.)
3. For each question show all of your work and **write legibly**. **Clearly indicate** your answers. No credit for illegible answers.
4. After the last page there is paper for scratch work. If you need extra scratch paper **after** you have filled these areas up, please raise your hand. Scratch paper must be turned in with your exam, with your name and ID number written on it, but scratch paper **will not** be graded.
5. Please write out the following statement: "*I pledge on my honor that I will not give or receive any unauthorized assistance on this examination.*"

6. Fill in the following:

NAME :
SIGNATURE :
SID :
SECTION NUMBER :

1. (20 points) Show that $(\frac{2}{3})^{1/5}$ is irrational. You can use either the MOD method (include proofs of all lemmas used) or the Unique Factorization method (you CAN assume all naturals factor uniquely into primes).

**SOLUTION TO PROBLEM ONE**

Assume, BWOC, that there exists $a, b$ in lowest terms such that

$$\left(\frac{2}{3}\right)^{1/5} = \frac{a}{b}$$

$$\frac{2}{3} = \frac{a^5}{b^5}$$

$$2b^5 = 3a^5$$

PROOF ONE:

2 divides $3a^5$. Hence 2 divides $a^5$.

Now need Lemma. Blah Blah. The rest omitted.

PROOF TWO:

Write $b$ and $a$ as prime factorization. Blah Blah.

**COMMENTS ON POINTS:**

If you did it using MOD stuff and got the proof of the helper lemma false, -10.

If there are things that you didn't explain in the proof (such as making claims that are non-trivial and-or not seen in class or HW without proof) -5 points.

If nothing right then -20 points.

2. (20 points- 6-7-7) For each of the following statements say if it is TRUE or FALSE. Give a BRIEF explanation. Put the TRUE or FALSE where we say. Put the BRIEF EXPLANATION where we say.

   There are three questions, the first one on this page, the next two on the next two pages.

   (a) If $A$ is a set whose powerset has 5 elements then $A$ is infinite.
   TRUE OR FALSE:
   BRIEF EXPLANATION:

(b) There exists a Boolean formula on EXACTLY 2 variables that has 4 satisfying assignments.

TRUE OR FALSE:

BRIEF EXPLANATION:

(c) There is a bijection from $\mathsf{Q}$ to $\mathsf{Q}^{\geq 0}$. ($\mathsf{Q}^{\geq 0}$ is the set of all rationals that are $\geq 0$.)
TRUE OR FALSE:
BRIEF EXPLANATION:

**SOLUTION TO PROBLEM TWO**

(a) If $A$ is a set whose powerset has 5 elements then $A$ is infinite.

TRUE – there are no such sets $A$, so true vacuously.

(b) There exists a Boolean formula on EXACTLY 2 variables that has 4 satisfying assignments.

TRUE- just take $(x \vee \neg x \vee y)$.

(c) There is a bijection from $\mathsf{Q}$ to $\mathsf{Q}^{\geq 0}$.

TRUE-

$\mathsf{Q}^{\geq 0}$ is countable:

List it out by listing out the elements in groups based on the sum of num and denom.

$\mathsf{Q}$ is countable: View this as $\{-1, 1\} \times \mathsf{Q}^{\geq 0}$.

Since both are countable:

There is a bijection $f : \mathsf{Q}$ to $\mathsf{N}$.

There is a bijection $g : \mathsf{N}$ to $\mathsf{Q}^{\geq 0}$.

Hence $g(f(x))$ is a bijection from $\mathsf{Q}$ to $\mathsf{Q}^{\geq 0}$.

3. (20 points 4-4-4-4-4) (PUT YOUR ANSWER ON THE NEXT PAGE.) Let $r, s \geq 5$. Assume that there is a deck of cards. Here are our rules:

   - Every card has a rank: a number in $\{1, \ldots, r\}$
   - Every card has a suit: a symbol in $\{S_1, \ldots, S_s\}$.
   - A *hand* is 4 cards.
   - A *Straight* is 4 cards with consecutive ranks, allowing wraparound. For example $(r-1, S_1)$, $(r, S_1)$, $(1, S_2)$, $(2, S_3)$. This DOES NOT include the case where all suits are the same.
   - A *Flush* is 4 cards of the same suit. For example $(1, S_1)$, $(3, S_1)$, $(4, S_1)$, $(r, S_1)$. This DOES NOT include the case where the hand is also a straight.
   - A *Straight Flush* is 4 cards that are both a straight and a flush. For example $(3, S_1)$, $(4, S_1)$, $(5, S_1)$, $(6, S_1)$.

   Suppose four cards are chosen at random.

   (a) What is the probability of getting a straight flush?

   (b) What is the probability of getting a straight? (DON'T FORGET- can't be a flush)

   (c) What is the probability of getting a flush? (DON'T FORGET- can't be a straight)

   (d) Give an ordered pair $(r, s)$ with $r, s \geq 5$ such that if the cards have $r$ ranks and $s$ suits then the prob of a straight is GREATER than the prob of getting a flush. (DON'T FORGET that a straight does not include a straight flush and a flush does not include a straight flush)

   (e) Give an ordered pair $(r, s)$ with $r, s \geq 5$ such that if the cards have $r$ ranks and $s$ suits then the prob of a flush is GREATER than the prob of getting a straight. (DON'T FORGET that a straight does not include a straight flush and a flush does not include a straight flush)

ANSWER PROBLEM THREE HERE:

**SOLUTION TO PROBLEM THREE**

The number of total hands is $\binom{rs}{4}$.

(a) What is the probability of getting a straight flush? A straight flush is determined by the first card's rank AND the suit. So thats $rs$.

(b) What is the probability of getting a straight? A straight is determined by the first card's rank and then 4 suits. So thats $rs^4$. Hence the prob, removing straight flushes, is

$$\frac{rs^4 - rs}{\binom{rs}{4}}.$$

(c) What is the probability of getting a flush? A flush is determined by the suit and then 4 cards of that suit, so thats $s\binom{r}{4}$. Hence the prob, removing straight flushes, is

$$\frac{s\binom{r}{4} - rs}{\binom{rs}{4}}.$$

(d) Give an ordered pair $(r, s)$ with $r, s \geq 5$ such that if the cards have $r$ ranks and $s$ suits then the prob of a straight is GREATER than the prob of getting a flush.

Need

$$s\binom{r}{4} - rs < rs^4$$

$$\binom{r}{4} - r < rs^3$$

$$\frac{r!}{4!(r-4)!} - r < rs^3$$

$$\frac{r(r-1)(r-2)(r-3)}{4!} - r < rs^3$$

$$\frac{(r-1)(r-2)(r-3)}{4!} - 1 < s^3$$

Suffices to make

$$r^3 - 1 < s^3$$

So just take $r = 5$ and $s = 6$.

(e)

$$s\binom{r}{4} - rs > rs^4$$

$$\binom{r}{4} - r > rs^3$$

$$\frac{r!}{4!(r-4)!} - r > rs^3$$

$$\frac{(r-1)(r-2)(r-3)}{4!} - 1 > s^3$$

$$\frac{(r-3)^3}{4!} - 1 > s^3$$

$$\frac{(r-3)^3}{24} - 1 > s^3$$

$$(r-3)^3 - 24 > 24s^3$$

$$(r-3)^3 > 24s^3 + 24$$

Take $r = 27$ and $s = 5$, which is overkill but does it:

$$24^3 > 24 \times 5^3 + 24$$

Divide by 24

$$24^2 > 5^3 + 1$$

which is clearly true.

**COMMENT ON POINTS:** If your answer to $a, b, c$ is wrong then we grade $d, e$ relative to your answers to $a, b, c$.

4. (20 points) (You can do this problem on this page AND if you need, the next page.) Let $T$ be defined by

   $T(1) = 5$

   $T(n) = 10T(\lfloor n^{1/3} \rfloor) + n^2$

   Use Constructive Induction to find $A, B \in \mathsf{N}$ such that

$$(\forall n \geq 1)[T(n) \leq n^A + B].$$

   Show all work. (HINT- to make your life easier DO NOT try to make $A$ or $B$ as small as possible.)

   You an use this page and the next page

   **SOLUTION TO PROBLEM FOUR**

   **Base Case:** $T(1) = 5$ so need $5 = T(1) \leq 1^A + B = 1 + B$. So $B \geq 4$.

   **IH:** For all $n' < n$, $T(n') \leq (n')^A + B$

   **IS:** We can assume $n \geq 2$.

$$T(n) = 10T(\lfloor n^{1/3} \rfloor) + n^2 \leq 10(n^{A/3} + B) + n^2 = 10n^{A/3} + 10B + n^2$$

   NEED

$$10n^{A/3} + 10B + n^2 \leq n^A + B$$

$$10n^{A/3} + 9B + n^2 \leq n^A$$

   Take $B = 4$

$$10n^{A/3} + 36 + n^2 \leq n^A$$

   The bigger $n$ is the easier it is to make this work. So take $n = 2$

$$10 \times 2^{A/3} + 36 + 2^2 \leq 2^A$$

   Since need $2^A \geq 36$ we try $A = 6$.

$$10 \times 2^2 + 36 + 2^2 \leq 2^6$$

OH- not quite. Try $A = 9$ (to avoid nasty fractions).

$$10n^3 + 36 + n^2 \leq n^9$$

Plug in $n = 2$ to get

$$10 \times 2^3 + 36 + 2^2 \leq 2^9$$

$$80 + 8 + 36 + 4 \leq 512$$

YES!

NOTE- $n = 7$ also works.

5. (20 points—5 points for the Y/N and 15 for the proof) (You can do this problem on this page AND if you need, the next page.) A function $f$ with domain $\mathsf{N}$ and co-domain $\mathsf{N}$ is GASARCHIAN if past some point $f$ is constant. Formally, with all quantifiers ranging over $\mathsf{N}$,

$$(\exists n_0, n_1 \in \mathsf{N})(\forall n \geq n_0)[f(n) = n_1].$$

Is the set of GASARCHIAN functions countable?

Put Y or N here:

Put your proof here (and can use the next page):

(You may use any theorem that was proven in class or the HW.)

**SOLUTION TO QUESTION FIVE**

YES the set of GASARCHIAN functions is countable.

If $f$ is such a function the we can identify it with a FINITE sequence of numbers, representing $f(0)$, $f(1)$, up to some finite $k$, so $f(k)$, and then ONE more number for what $f(k+1), f(k+2)$, etc is.

So there is a bijection from the set of Gasarchian Functions to

$$\mathsf{N} \cup \mathsf{N} \times \mathsf{N} \cup \mathsf{N} \times N \times \mathsf{N} \cup$$

Each of these is countable since any finite cross product of countable sets is countable.

The union is countable since the union of countable sets is countable.

**COMMENT ON POINTS:**

If you said UNCOUNTABLE then 0 points.

If you said COUNTABLE but nonsense proof 5 points.

Some students had the following incorrect but good-idea proof: The set of GASARCHIAN functions is the same as $N \times N$ since you map the function $f$ to $(n_0, n_1)$.

NO GOOD: Many functions map to $(n_0, n_1)$.

Some students did it the other way around: map $(n_0, n_1)$ to the function. But there are many functions.

That proof COULD be made correct by saying that

$(n_0, n_1)$ maps to the SET of all functions with those parameters. Call that SET $X_{n_0,n_1}$. Then show that each $X_{n_0,n_1}$ is COUNTABLE, and use UNION of countable sets is countable.