Homework 3, MORALLY Due Feb 18

WARNING: THIS HW IS TWO PAGES LONG!!!!!!!!!!!!!!!!!

1. (40 points)

   (a) (20 points) We are working in binary so all numbers are 0's and 1's. When we input 2 bits to a circuit we think of it as a NUMBER in base 2.

   00=0

   01=1

   10=2

   11=3

   The output will be 3 bits, intepreted as a number in base 2.

   000=0, . . ., 111=7

   Write a truth table with 2 inputs and 3 outputs for the following function:

   $f(xy) = (xy)^2 \pmod 8$.

   So for example

   $f(11)$ is computed by 11=3, $3^2 = 9$, $9 \pmod 8 = 1$, so the output is 001.

   (b) (20 points) Write a circuit for f using AND, OR, and NOT using the method shown in class (do not simplify- that would make it harder for the TA's to grade!)

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2. (60 points) In this problem the inputs are considered bits that you add. So if the input is $(1, 0, 1)$ it is NOT 101 in base 2 which is 5. Its just three bits, separate.

   The *Depth* of a circuit is the max number of gates from input to output.

   The *Size* of a circuit is the total number of gates.

   ALSO- we allow for input the variables AND their negations.

   In this problem we will look at different circuits for:

$$f_n(x_1, \ldots, x_n) = x_1 + \cdots + x_n \pmod 2.$$

   We allow AND, OR and NOT gates usual; however, the AND and OR gates can take MANY inputs (as many as you want).

   You can assume that n is a power of two if it makes the math easier.

   (a) (30 points) Show that, for all n, there is a circuit for f_n with depth 2. What is the circuit's size?

   (b) (30 points) Show that, for all n, there is a circuit for f_n with size $O(n)$ (less than some constant times n). What is the constant? What is the depth of the circuit? (HINT: First get a circuit for $f_2(x_1, x_2)$. View this as a gate you can use. Use that

$$f_n(x_1, \ldots, x_n) = f_{n/2}(f_2(x_1, x_2), f(x_3, x_4), \ldots, f(x_{n-1}, x_n)).$$

   (c) (0 points but think about) In part 1 you got a CONSTANT DEPTH but LARGE SIZE circuit. In part 2 you got a SMALL SIZE but LOG DEPTH circuit. Is there a circuit for f_n which is constant depth and small size?