Homework 5, MORALLY Due March 25

WARNING: THIS HW IS TWO PAGES LONG!!!!!!!!!!!!!!!!!

1. (20 points)

   (a) (0 points) What is $\int_1^n x^2 dx$?

   (b) (2 points) Use the answer to part 1 to conjecture the FORM of the formula (with some of the coefficients not know) for

   $$\sum_{i=1}^{n} i^2$$

   (To make your life easier you can also conjecture what the first coefficient is based on the interval.)

   (c) (9 points) (Use part b) By plugging in $n = 0$, $n = 1$, and perhaps more find a very good guess for the formula for $\sum_{i=1}^{n} i^2$. Show your work of course. (It should be the real formula.)

   (d) (9 points) (Use part b) Derive the formula by constructive induction.

2. (20 points) Recall our usual *induction scheme*:

   From

   - $P(0)$
   - $(\forall n \ge 0, n \in \mathsf{N})[P(n) \rightarrow P(n+1)]$

   we get $(\forall n \in \mathsf{N})[P(n)]$.

   This problem is about how to modify this scheme.

   (a) (7 points) Give a scheme that will show

   $$(\forall n \equiv 0 \pmod 4, n \in \mathsf{N})[P(n)].$$

   (b) (7 points) Give a scheme that will show

   $$(\forall n \equiv 0, 1 \pmod 4, n \in \mathsf{N})[P(n)].$$

   (c) (6 points) Give a scheme that will show

   $$(\forall n \in \mathsf{Z})[P(n)].$$

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3. (20 points) Assume that there are constants A, B, C, D such that

$$(\forall n \geq 0, n \in \mathsf{N}) \left[\sum_{i=1}^{n} i \times 2^i = An2^n + B2^n + Cn + D \right].$$

   (a) (10 points) Find A, B, C, D by plugging in $n = 0, 1, 2, 3$ (or less if you don't need all of those) into the equation.
   (b) (10 points) Find A, B, C, D by constructive induction.
   (c) (0 points- don't hand in) What are the PROS and CONS of each technique?
   (d) (0 points- don't hand in) How could you have guessed the form of the summation above? One way is with integrals. Can you think of another way?

4. (20 points) Let $T(n)$ be defined by

$$T(1) = 10$$

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + 17n$$

   By constructive induction find value $c \in \mathsf{N}$ such that $(\forall n)[T(n) \leq cn$. Try to make c as small as possible (and its in N so this is possible).

5. (20 points) In the country of Fredonia they only use 10-cent coins and 11-cent coins. Note that the people cannot have 9 cents on them, they can have 10, they can have 11, but they can't have 12.

   (a) (0 points and don't hand anything in) Write a program that will, for $n = 1$ to 1000, determine which numbers of cents good people of Fredonia can have.
   (b) (5 points) Make a conjecture of the form:
      - $n_0 - 1$ CANNOT be written in the form $10x + 11y$ with $x, y \in \mathsf{N}$.
      - $(\forall n \geq n_0)(\exists x, y \in \mathsf{N})[n = 10x + 11y]$.

      (So you need to find n_0.)
   (c) (15 points) Prove your conjecture by induction.