Homework 7
Morally due FRI Apr 5, 5:00PM. DEAD CAT Monday Apr 8 5:00
(NOTE- Change of when its due is so that we can go over it in rec
BEFORE the exam.)

THE HW IS TWO PAGES LONG!!!!!!!!!!!!!!

1. (20 points)

(a) (10 points) WG, Jtwitty, and K are taking the class on a field trip
to the Combinatorics Museum! There are 32 students in the class
WG will drive 18 of them.
Jtwitty will drive 7 of them.
K will drive 7 of them.

How many ways can the students choose which cars they want to
be in?

(b) (15 points) Generalize the problem as follows. $A_1, \ldots, A_n$ are tak-
ing the class on a field trip! There are $S$ students in the class.
$A_1$ will drive $a_1$ of them.

:\
$A_n$ will drive $a_n$ of them.
(Note that $a_1 + \cdots + a_n = S$.)

How many ways can the students choose which cars they want to
be in?

SOLUTION TO QUESTION 1

1a) We do it two ways

18 students are chosen for WG, so that $\binom{32}{18}$. Note that there are 14
students left.

7 students of those left are chosen for Jtwitty, so that $\binom{14}{7}$.

The rest go to K.

So the answer is $\binom{32}{18} \times \binom{14}{7}$.

This is a fine answer; however note that it is also $\frac{32!}{18!14!}$ which leads to
the second method.

We view this as the number of ways of arranging 32 students- the first
18 go with WG, the next 7 go with Jtwitty, the next 7 go with K. BUT
we then want to NOT CARE about the order WITHIN the first 18, WITHIN the next 7, WITHIN the next 7. So that's directly \( \frac{32}{18!7!7!} \).

1b) \( \left( \frac{s!}{a!b!c!} \right) \).

2. (25 points) Use a combinatorial argument (NOT algebraic, NOT by induction) to show that if \( S = a + b + c \) then

\[
\frac{S!}{a!b!c!} = \frac{(S - 1)!}{(a - 1)!b!c!} + \frac{(S - 1)!}{a!(b - 1)!c!} + \frac{(S - 1)!}{a!b!(c - 1)!}
\]

**SOLUTION TO QUESTION 2**

The number of ways of assigning \( a \) students to WG, \( b \) students to K, \( c \) students to Jtwitty is

\[
\frac{S!}{a!b!c!}
\]

We can break this up into three disjoint sets:

Let Alice be a student in the class.

The number of ways they can do this where WG drives Alice is

\[
\frac{(S - 1)!}{(a - 1)!b!c!}
\]

The number of ways they can do this where K drives Alice is

\[
\frac{(S - 1)!}{a!(b - 1)!c!}
\]

The number of ways they can do this where Jtwitty drives Alice is

\[
\frac{(S - 1)!}{a!b!(c - 1)!}
\]

Add them up for the proof!

**GOTO NEXT PAGE**
3. (25 points) Fill in the blanks in the following statement. Describe your reasoning. BLANK will be a function of \( k, n \).

If \( A \subseteq \{1, \ldots, n\} \) and \( |A| = k \) then at least BLANK subsets of \( A \) have the same SUM.

**SOLUTION TO QUESTION 3**

There are \( 2^k \) subsets of \( A \).

The min sum is 0.

The max sum is \( n + (n - 1) + \cdots + (n - k + 1) \) which is

\[
\sum_{i=1}^{n} - \sum_{i=1}^{k-1} i = 1^{n-k} = \frac{n(n + 1) - (n - k)(n - k + 1)}{2}
\]

Numerator is:

\[
n^2 + n - (n^2 - 2kn + n - k + k^2) = n^2 + n - n^2 + 2kn - n + k - k^2
\]

\[
= 2kn + k - k^2
\]

So the max sum is \( \frac{2kn + k - k^2}{2} \).

Hence the total number of sums is \( \frac{2kn + k - k^2 + 2}{2} \).

Hence the number of sets that have the same sum is at least

\[
\left\lfloor \frac{2^k}{(2kn + k - k^2 + 2)/2} \right\rfloor = \left\lfloor \frac{2^{k+1}}{2kn + k - k^2 + 2} \right\rfloor
\]

4. (25 points) Show that no matter how you 3-color the \( 4 \times 19 \) grid there will be a monochromatic rectangle.

**SOLUTION TO PROBLEM 4**

Let \( COL \) be a 3-coloring of the \( 4 \times 19 \) grid.

Note that every column has:

- Some pair \( 1 \leq i < j \leq 4 \) such that the \( i \)th and \( j \)th entry in the column have the same color.
• That color which we call $c$.

MAP every column to $(\{i, j\}, c)$.
The columns are the balls. There are 19 of them.
The elements of $\binom{\{1, 2, 3, 4\}}{2} \times \{R, W, B\}$ are the boxes. There are $\binom{4}{2} \times 3 = 6 \times 3 = 18$ boxes.