Homework 7

Morally due FRI Apr 5, 5:00PM. DEAD CAT Monday Apr 8 5:00

(NOTE- Change of when its due is so that we can go over it in rec BEFORE the exam.)

THE HW IS TWO PAGES LONG!!!!!!!!!!!!!

1. (20 points)

   (a) (10 points) WG, Jtwitty, and K are taking the class on a field trip to the Combinatorics Museum! There are 32 students in the class

   WG will drive 18 of them.

   Jtwitty will drive 7 of them.

   K will drive 7 of them.

   How many ways can the students choose which cars they want to be in?

   (b) (15 points) Generalize the problem as follows. $A_1, \ldots, A_n$ are taking the class on a field trip! There are S students in the class.

   A_1 will drive a_1 of them.

   $\vdots$

   A_n will drive a_n of them.

   (Note that $a_1 + \cdots + a_n = S$.)

   How many ways can the students choose which cars they want to be in?

SOLUTION TO QUESTION 1

1a) We do it two ways

18 students are chosen for WG, so that $\binom{32}{18}$. Note that there are 14 students left.

7 students of those left are chosen for Jtwitty, so that $\binom{14}{7}$.

The rest go to K.

So the answer is $\binom{32}{18} \times \binom{14}{7}$.

This is a fine answer; however note that it is also $\frac{32}{18!7!7!}$ which leads to the second method.

We view this as the number of ways of arranging 32 students- the first 18 go with WG, the next 7 go with Jtwitty, the next 7 go with K. BUT we then want to NOT CARE about the order WITHIN the first 18, WITHIN the next 7, WITHIN the next 7. So thats directly $\frac{32}{18!7!7!}$

1b) $\binom{S!}{a!b!c!}$.

2. (25 points) Use a combinatorial argument (NOT algebraic, NOT by induction) to show that if $S = a + b + c$ then

$$\frac{S!}{a!b!c!} = \frac{(S-1)!}{(a-1)!b!c!} + \frac{(S-1)!}{a!(b-1)!c!} + \frac{(S-1)!}{a!b!(c-1)!}$$

SOLUTION TO QUESTION 2

The number of ways of assigning a students to WG, b students to K, c students to Jtwitty is

$$\frac{S!}{a!b!c!}.$$

We can break this up into three disjoint sets:

Let Alice be a student in the class.

The number of ways they can do this where WG drives Alice is

$$\frac{(S-1)!}{(a-1)!b!c!}.$$

The number of ways they can do this where K drives Alice is

$$\frac{(S-1)!}{(a-1)!b!c!}.$$

The number of ways they can do this where Jtwitty drives Alice is

$$\frac{(S-1)!}{a!b!(c-1)!}.$$

Add them up for the proof!

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3. (25 points) Fill in the blanks in the following statement. Describe your reasoning. BLANK will be a function of k, n.

   If $A \subseteq \{1, \ldots, n\}$ and $|A| = k$ then at least BLANK subsets of A have the same SUM.

   SOLUTION TO QUESTION 3

   There are 2^k subsets of A.

   The min sum is 0

   The max sum is $n + (n-1) + \cdots + (n-k+1)$ which is

$$\sum_{i=1}^{n} - \sum i = 1^{n-k} = \frac{n(n+1) - (n-k)(n-k+1)}{2}$$

   Numerator is:

$$n^2 + n - (n^2 - 2kn + n - k + k^2) = n^2 + n - n^2 + 2kn - n + k - k^2$$

$$= 2kn + k - k^2$$

   So the max sum is $\frac{2kn+k-k^2}{2}$.

   Hence the total number of sums is $\frac{2kn+k-k^2+2}{2}$.

   Hence the number of sets that have the same sum is at least

$$\left\lceil \frac{2^k}{(2kn + k - k^2 + 2)/2} \right\rceil = \left\lceil \frac{2^{k+1}}{2kn + k - k^2 + 2} \right\rceil$$

4. (25 points) Show that no matter how you 3-color the 4×19 grid there will be a monochromatic rectangle.

   SOLUTION TO PROBLEM 4

   Let COL be a 3-coloring of the 4×19 grid.

   Note that every column has:

   - Some pair $1 \leq i < j \leq 4$ such that the ith and jth entry in the column have the same color.

- That color which we call c.

MAP every column to $(\{i, j\}, c)$.

The columns are the balls. There are 19 of them.

The elements of $\binom{\{1,2,3,4\}}{2} \times \{R, W, B\}$ are the boxes. There are $\binom{4}{2} \times 3 = 6 \times 3 = 18$ boxes.