250 MIDTERM

Do not open this exam until you are told. Read these instructions:

1. This is a closed book exam, though ONE sheet of notes is allowed. **No calculators, or other aids are allowed**. If you have a question during the exam, please raise your hand.
2. There are 5 problems which add up to 100 points. The exam is 1 hours 15 minutes. (You shouldn't need that much.)
3. For each question show all of your work and **write legibly**. **Clearly indicate** your answers. No credit for illegible answers.
4. After the last page there is paper for scratch work. If you need extra scratch paper **after** you have filled these areas up, please raise your hand. Scratch paper must be turned in with your exam, with your name and ID number written on it, but scratch paper **will not** be graded.
5. Please write out the following statement: "*I pledge on my honor that I will not give or receive any unauthorized assistance on this examination.*"

6. Fill in the following:

NAME :
SIGNATURE :
SID :
SECTION NUMBER :

SCORES ON PROBLEMS

Prob 1:	
Prob 2:	
Prob 3:	
Prob 4:	
Prob 5:	
TOTAL	

1. (20 points) For this problem, as usual TRUE is 1 and FALSE is 0.

   (a) Fill in the table below so that it is a truth table for the function:

$$f(x, y, z) = xy + z \pmod 2$$

x	y	z	$f(x, y, z)$
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

   (b) Write down a Boolean Formula for f.

SOLUTION TO PROBLEM ONE

a)

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

b) Omitted.

2. (20 points) Assume Unique Factorization. Show that if p and q are DISTINCT primes then $(pq)^{2/7} \notin \mathsf{Q}$.

SOLUTION TO PROBLEM TWO

Assume, by way of contradiction, that there exists $a, b \in \mathsf{Z}$ such that

$(pq)^{2/7} = \frac{a}{b}$

$b(pq)^{2/7} = a$

$b^7 p^2 q^2 = a^7$

Write $a = \prod_{i=1}^{L} p_i^{a_i}$

Write $b = \prod_{i=1}^{L} p_i^{b_i}$

By renumbering assume that $p_1 = p$ and $p_2 = q$. Then

Write $a = p^{a_1} q^{a_2} \prod_{i=3}^{L} p_i^{a_i}$

Write $b = p^{b_1} q^{b_2} \prod_{i=3}^{L} p_i^{b_i}$

SO now we have

Write $a^7 = p^{7a_1} q^{7a_2} \prod_{i=3}^{L} p_i^{7a_i}$

Write $b^7 = p^{7b_1} q^{7b_2} \prod_{i=3}^{L} p_i^{7b_i}$

From $b^7 p^2 q^2 = a^7$ we have

$$p^{7b_1} q^{7b_2} \prod_{i=3}^{L} p_i^{7b_i} p^2 q^2 = p^{7a_1} q^{7a_2} \prod_{i=3}^{L} p_i^{7a_i}$$

$$p^{7b_1+2} q^{7b_2+2} \prod_{i=3}^{L} p_i^{7b_i} p^2 q^2 = p^{7a_1} q^{7a_2} \prod_{i=3}^{L} p_i^{7a_i}$$

So we have $7b_1 + 2 = 7a_1$ and $7b_2 + 2 = 7a_2$. Take this mod 7 to get

$$2 \equiv 0 \pmod 7$$

This is a contradiction

COMMENT ON WHAT YOU DID WRONG

Some students got to this step:

$$p^{7b_1+2}q^{7b_2+2}\prod_{i=3}^{L} p_i^{7b_i} p^2 q^2 = p^{7a_1}q^{7a_2}\prod_{i=3}^{L} p_i^{7a_i}$$

and said its a contradiction. That is incomplete (and lost five points).

Some students got it to $7b_2 + 2 = 7a_2$ but did not say why its a contradiction. They really should do it mod 7 as above, but this we did not penalize for. We will in the future.

3. (20 points)

   (a) Give an example of a Boolean Formula on 5 variables that has exactly TWO satisfying assignments. Keep your formulas SIMPLE! If we can't easily see that your formula has TWO satisfiying assignments you will get a zero. (You must use all five varibles.)

   (b) Give an example of a Boolean Formula on n variables that has exactly TWO satisfying assignments. Keep your formulas SIMPLE! If we can't easily see that your formula has TWO satisfiying assignments you will get a zero. (You must use all n variables.)

SOLUTION TO PROBLEM THREE

(a) $(x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge x_5) \vee (x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5)$.

(b) $(x_1 \wedge \cdots x_2 \cdots \wedge x_n) \vee (x_1 \wedge \cdots \wedge x_{n-1} \wedge \neg x_n)$.

4. (20 points) Recall that R is the reals.

   Give an infinite domain $D \subseteq \mathsf{R}$ where BOTH of the following holds and JUSTIFY your answer.

   - The statement

     $$(\forall x \in D)(\forall y \in D)[x < y \rightarrow (\exists z \in D)[x < z < y]$$

     is TRUE, and
   - The statement
     $$(\forall x \in D)(\exists y \in D)[y = 2x]$$
     is FALSE.

   SOLUTION TO PROBLEM FOUR

   We give an infinite domain where

   - The statement $(\forall x)(\forall y)[x < y \rightarrow (\exists z)[x < z < y]$ is TRUE, and
   - The statement $(\forall x)(\exists y)[y = 2x]$ is FALSE.

   Justify your answer.

   Domain $[0, 1]$, all the reals between 0 and 1. First statement true because given x, y can take $z = (x + y)/2$.

   Second statement false since can take $x = 1$. The only y that can work is $y = 2$ which is not in the domain.

   COMMENTS ON WHAT YOU DID WRONG

   Some students didn't give a domain D. The problem asked for a domain D so you really do have to give one.

   Some students Domains satisfied only one of the conditions. They got 10 points.

   Some student gave the answer D is the irrationals. This does not work. The statement

   $(\forall x)(\exists y)[y = 2x]$ is TRUE: we leave it to you to prove that

   If x is IRRATIONAL then $2x$ is IRRATIONAL.

5. (20 points) For each of the following statements say if it is TRUE or FALSE and justify your answer. The DOMAIN is Z (the integers). You CAN use that $\sqrt{2}, \sqrt{7}, \sqrt{8} \notin \mathsf{Q}$ but no other number being irrational. (The expression $\mathsf{Z} - \{0\}$ is the set of NONZERO integers.)

   (a) $(\exists x \in \mathsf{Z})(\exists y \in \mathsf{Z})[x\sqrt{2} + y\sqrt{7} \in \mathsf{Z} - \{0\}]$.

   (b) $(\exists x \in \mathsf{Z})(\exists y \in \mathsf{Z})[x\sqrt{2} + y\sqrt{8} \in \mathsf{Z} - \{0\}]$.

SOLUTION TO PROBLEM FIVE

(a) $(\exists x)(\exists y)[x\sqrt{2} + y\sqrt{7} \in \mathsf{Z}]$.
FALSE: Proof by contradiction. If
$x\sqrt{2} + y\sqrt{7} = z$ where $z \in \mathsf{Z} - \{0\}$.
If $y = 0$ then we get $x\sqrt{2} = z$ so $\sqrt{2} = z/x$, a contradiction. Hence we can assume $y \neq 0$.
$x\sqrt{2} = z - y\sqrt{7}$
$2x^2 = z^2 - 2yz\sqrt{7} + 7$
Can get $\sqrt{7}$ as a rational from this (need $yz \neq 0$ do can divide by it). CONTRADICTION.

COMMENTS ON WHAT YOU DID

Many students thought that if α, β are IRRATIONAL then $\alpha + \beta$ is IRRATIONAL. This is not true. Note that

$$\sqrt{2} - \sqrt{2} = 0$$

AH- you are thinking OKAY, ITS EITHER 0 OR IRRATIONAL. NOPE

$$(1 - \sqrt{2}) + \sqrt{2} = 1$$

is the sum of two irrationals that is rational.

(b) $(\exists x)(\exists y)[x\sqrt{2} + y\sqrt{8} \in \mathsf{Q}]$.
FALSE:
$\sqrt{8} = 2\sqrt{2}$, so we need x, y such that
$x\sqrt{2} + 2y\sqrt{2} \in \mathsf{Z} - \{0\}$
$(x + 2y)\sqrt{2} = z$ where $c \in Z - \{0\}$.

Since $z \neq 0$, $x + 2y \neq 0$. Hence you can divide by it.
Hence we have $\sqrt{2} = \frac{z}{x+2y}$
This is a contradiction.

WHAT SOME STUDENTS DID WRONG

It is crucial that you us that $z \neq 0$. If a student did not use this then he was marked wrong. Statements like

A rational times and irrational is always irrational

is not quite right since the rational could be 0.

Scratch Paper