Homework 1

Morally due Mon Feb 8, 9:00AM. DEAD CAT Wed Feb 10 9:00AM

THE HW IS FIVE PAGES LONG!!!!!!!!!!!!!

1. (20 points) You can use Wolfram Alpha on this problem for the calculations. In the year 2030 there are 30 students taking CMSC 250H. There are three TAs: Alice, Bob, and Carol. The students are broken into three project groups, each lead by one of the TA's.

   (a) (6 points) Assume Alice will take 10 students, Bob will take 10 students, Carol will take 10 students.

   How many ways can the students be assigned to projects? Explain your answer. Give the number in terms of BOTH factorials and as an actual number.

   (b) (7 points) Assume Alice will take 5 students, Bob will take 10 students, Carol will take 15 students.

   How many ways can the students be assigned to projects? Explain your answer. Give the number in terms of BOTH factorials and as an actual number.

   (c) (7 points) In the year 2040 there are n students taking CMSC 250H. There are k TAs: $A_1, \ldots, A_k$. The students are broken into k project groups, each lead by one of the TA's.

   Assume A_1 will take n_1 students, ..., A_k will take n_k students (Note that $n_1 + \cdots + n_k = n$.)

   How many ways can the students be assigned to projects?

   (d) (0 points but DO IT anyway since we are all here to learn) Speculate: What should $n_1, \ldots, n_k$ be to maximize the number of ways students can be assigned to projects?

SOLUTION

1a) We do it two ways

FIRST WAY:

10 students are chosen for Alice, so that $\binom{30}{10}$. Note that there are 14 students left.

20 students of those left are chosen for Bob, so that $\binom{20}{10}$.

The rest go to Carol.

So the answer is $\binom{30}{10} \times \binom{20}{10}$.

$$\binom{30}{10} \times \binom{20}{10} = \frac{30!}{20!10!}\frac{20!}{10!10!} = \frac{30!}{10!}\frac{1}{10!10!}$$

$$= \left(\frac{30!}{10!10!10!}\right) = \frac{30 \times \cdots \times 11 \times 10!}{10!10!10!}$$

$$= \frac{30 \times \cdots \times 11}{10!10!}$$

Using Wolfram Alpha I found that this is 5,550,996,791,340.

SECOND WAY

We view this as the number of ways of arranging 30 students- the first 10 go with Alice, the next 10 go with Bob, the next 10 go with Carol. BUT we then want to NOT CARE about the order WITHIN the first 10, WITHIN the next 10, WITHIN the next 10. So thats directly $\frac{30}{10!10!10!}$

1b) $\frac{30!}{5!10!15!}$.

Using Wolfram Alpha I found that this is 465,817,912,560.

1c) $\frac{n!}{n_1!\cdots n_k!}$.

1d) Maximized when the n_i are as equal as possible.

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2. (20 points) Use a combinatorial argument (NOT algebraic) to show that if $S = a + b + c$ then

$$\frac{S!}{a!b!c!} = \frac{(S-1)!}{(a-1)!b!c!} + \frac{(S-1)!}{a!(b-1)!c!} + \frac{(S-1)!}{a!b!(c-1)!}$$

SOLUTION TO QUESTION 2

If there are S students and we assign a to Alice, b to Bob, and c to Carol then how many ways can we do that? We solve this two ways and these two will be equal.

WAY ONE: $\frac{S!}{a!b!c!}$, as shown in the last problem.

WAY TWO: We can break this up into three disjoint sets:

Let Donna be a student in the class.

The number of ways they can do this where Donna is in Alice's group is

$$\frac{(S-1)!}{(a-1)!b!c!}.$$

The number of ways they can do this where Donna is in Bob's group is

$$\frac{(S-1)!}{(a-1)!b!c!}.$$

The number of ways they can do this where Donna drives Alice is

$$\frac{(S-1)!}{a!b!(c-1)!}.$$

Add them up for the proof!

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3. (20 points)

(a) (10 points) Fill in the blanks in the following statement. Describe your reasoning. BLANK will be a function of k, n.

If $A \subseteq \{1, \ldots, n\}$ and $|A| = k$ then at least BLANK subsets of A have the same SUM.

(b) (10 points) Write a program that, on input k, n determines (1) the most number of subsets of size k that have the same sum which we denote m, and (2) that sum which we denote s.

EXAMPLE: If $n = 4$ and $k = 2$ then

$1 + 2 = 3$

$1 + 3 = 4$

$1 + 4 = 5$

$2 + 3 = 5$

$2 + 4 = 6$

$3 + 4 = 7$.

Note that 5 occurs twice and everything else occurs once, so $m = 2$ and $s = 5$.

For $n = 10$ and $k = 2, \ldots, 9$ run your program and present a table of the following information.

k	q	m	$m - q$	s
2				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
9				

SOLUTION TO QUESTION 3

3a)

There are 2^k subsets of A.

The min sum is 0

The max sum is $n + (n-1) + \cdots + (n-k+1)$ which is

$$\sum_{i=1}^{n} - \sum i = 1^{n-k} = \frac{n(n+1) - (n-k)(n-k+1)}{2}$$

Numerator is:

$$n^2 + n - (n^2 - 2kn + n - k + k^2) = n^2 + n - n^2 + 2kn - n + k - k^2$$

$$= 2kn + k - k^2$$

So the max sum is $\frac{2kn+k-k^2}{2}$.
Hence the total number of sums is $\frac{2kn+k-k^2+2}{2}$.
Hence the number of sets that have the same sum is at least

$$\left\lceil \frac{2^k}{(2kn+k-k^2+2)/2} \right\rceil = \left\lceil \frac{2^{k+1}}{2kn+k-k^2+2} \right\rceil$$

3b) Omitted.

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SOLUTION TO PROBLEM 4

Let COL be a 3-coloring of the 4×19 grid.

Note that every column has:

- Some pair $1 \le i < j \le 4$ such that the ith and jth entry in the column have the same color.
- That color which we call c.

MAP every column to $(\{i,j\},c)$.

The columns are the balls. There are 19 of them.

The elements of $\binom{\{1,2,3,4\}}{2} \times \{R,W,B\}$ are the boxes. There are $\binom{4}{2} \times 3 = 6 \times 3 = 18$ boxes.

4. (20 points-5 points each) Bill makes his glirand a lunch every day. It has

(1) Sandwich: egg salad OR peanut butter OR tuna fish OR turkey OR ham.

(2) Fruit: apple OR orange OR grapefruit OR coconut.

(3) Snack: applesauce OR pretzels OR cheese.

(4) Drink: apple juice OR beer.

And NOW finally the questions: Each question below is separate.

(a) How many ways can Bill make his glirand's lunch? **SOLUTION** $5 \times 4 \times 3 \times 2 = 120$

(b) glirand says *I do not want to have BOTH and Apple and Applesauce* NOW how many ways can Bill make his glirand's lunch?

SOLUTION We first find the number that DO have both Apple and Applesauce:

$5 \times 1 \times 1 \times 2 = 10$.

So the answer is $120 - 10 = 110$.

(c) glirand says *I WANT to have two of the three Apple-things, but NOT all three.* NOW how many ways can Bill make his glirand's lunch?

SOLUTION There are three cases:

Apple and Applesauce but NO apple juice: $5 \times 1 \times 1 \times 1 = 5$.

Apple and Apple juice but NO applesauce: $5 \times 1 \times 2 \times 1 = 10$.

Applesauce and Apple juice but NO apple: $5 \times 3 \times 1 \times 1 = 15$.

So the answer is $5 + 10 + 15 = 30$.

(d) glirand says *IF you give me a turkey or ham sandwich then you MUST give me a cheese snack.* NOW how many ways can Bill make his glirand's lunch? **SOLUTION** There are three cases.

Has Turkey: $1 \times 4 \times 1 \times 2 = 8$

Has Ham: $1 \times 4 \times 1 \times 2 = 8$

Has neither Turkey nor Ham: $3 \times 4 \times 3 \times 2 = 72$.

(e) (0 points) glirand is a permutation of what Bill really calls his wife. What does Bill call his wife?

glirand is a permutation of darling.

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5. (20 points)

   (a) Do this problem from first principles, not using the formula. What is the coefficient of $v^2w^2x^2y^2z^2$ in the expansion of

   $$(v + w + x + y + z)^{10}.$$

   (As a sanity check see that it agrees with the formula.)

   SOLUTIONS

   We give two solutions

   FIRST SOLUTION:

   Pick out the pair that contribute 2 v's: $\binom{10}{2}$.

   Pick out the pair that contribute 2 w's: $\binom{8}{2}$.

   Pick out the pair that contribute 2 x's: $\binom{6}{2}$.

   Pick out the pair that contribute 2 y's: $\binom{4}{2}$.

   NOW the pair that contribute 2 z's is determined. So the answer is

   $\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}$.

   SECOND SOLUTION

   If, say, the coefficient gets

   the v from the 1st and 3rd factor

   the w from the 2nd and 4th factor

   the x from the 5th and 6th factor

   the y from the 7th and 10th factor

   the z from the 8th and 9th factor

   We can represent that as $vwvwxxyzzy$.

   More generally, we can rep any way to get the term as a sequence of 2 v's, 2 w's, 2 x's, 2 y's, and 2 z's.

   So how many ways are there to form words with those letters?

   $\frac{10!}{2!2!2!2!2!}$.