Homework 02, Morally due Mon Feb 15, 9:00AM

Throughout this HW:

- Let $f(m,s)$ be the muffin function (from the talk Bill gave on Muffins).
- You CAN use the Floor-Ceiling Theorem. (Advice: Write a program for it.)

1. (0 points but if you miss the midterm that means you got this wrong retroactively and you will lose a lot of points). When is the midterm? By what day do you need to tell Dr. Gasarch that you cannot make the midterm (if you cannot and know ahead of time)?

2. (40 points-10 each) Prove the following: (You are allowed to say 'by the floor ceiling theorem' if its true, without any further elaboration.

   (a) $f(8,7) \le \frac{5}{14}$
   (b) $f(12,7) \le \frac{3}{7}$.
   (c) $f(18,7) \le \frac{3}{7}$.
   (d) $f(19,7) \le \frac{25}{56}$.

   a) $f(8,7) \le \frac{5}{14}$.

   SOLUTION

   Assume that there is a procedure that does BETTER THAN $\frac{5}{14}$.

   Case 1: Some muffin is cut into ≥ 3 pieces. Then some piece is $\le \frac{1}{3} < \frac{5}{14}$.

   Case 2: Some muffin is uncut. One can cut each uncut muffin 1/2-1/2 and give the two halves to the recipient.

   Case 3: All muffins are cut into exactly two pieces. So there are 16 pieces.

   Case 3a: Alice gets ≥ 4 shares. Then Alice has a piece $\le \frac{8}{7} \times \frac{1}{4} = \frac{2}{7} < \frac{5}{14}$.

   Case 3b: Alice gets ≤ 1 shares, so Alice gets 1 share. Alice gets $\frac{8}{7}$ so 1 share is not enough!

   Case 3c: Everyone gets 2 or 3 shares. Let s_2 (s_3) be the number of people who get 2 (3) shares.

$2s_2 + 3s_3 = 16$

$s_2 + s_3 = 7$

SO $s_2 = 5$ and $s_3 = 2$.

A student who gets 2 shares is called a 2-student (and 3-shares is called a 3-student). A share that goes to a 2-student is a 2-share. Note that there are 10 2-shares and 6 3-shares.

Claim: All 2-shares are $> \frac{1}{2}$.

Proof: Assume Alice has a 2-share $\leq \frac{1}{2}$. Then the other 2-share Alice has is $\geq \frac{8}{7} - \frac{1}{2} = \frac{9}{14}$. That piece's buddy is $\leq 1 - \frac{9}{14} = \frac{5}{14}$.

End of Proof of Claim

All 2-shares are $> \frac{1}{2}$. There are 10 2-shares. Hence there are 10 pieces that are $> \frac{1}{2}$. This is impossible since 8 muffins were cut into two pieces so there are at most 8 piece $> \frac{1}{2}$.

b) and c) are by Floor Ceiling.

d) $f(19, 7)$ is similar to the $f(8, 7) \leq \frac{5}{14}$ and hence omitted.

END OF SOLUTION

3. (30 points- 15 each) Show the following by giving a procedure.

   (a) $f(12, 7) \geq \frac{3}{7}$.

   (b) $f(18, 7) \geq \frac{3}{7}$.

SOLUTION

Procedure that shows $f(12, 7) \geq \frac{3}{7}$:

(a) Divide 12 muffins $\{\frac{3}{7}, \frac{4}{7}\}$.

(b) Give 3 students $\{\frac{4}{7}, \frac{4}{7}, \frac{4}{7}\}$.

(c) Give 4 students $\{\frac{3}{7}, \frac{3}{7}, \frac{3}{7}\}$.

Procedure that shows $f(18, 7) \geq \frac{3}{7}$:

(a) Divide 18 muffins $\{\frac{3}{7}, \frac{4}{7}\}$.

(b) Give 1 students $\{\frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}\}$.

(c) Give 6 students $\{\frac{3}{7}, \frac{3}{7}, \frac{4}{7}, \frac{4}{7}, \frac{4}{7}\}$.

4. (30 points) If you combine parts of problem 2 with all of problem 3 you get $f(12, 7) = \frac{3}{7}$ and $f(18, 7) = \frac{3}{7}$. FIND two more values of m (rel prime to 7) such that $f(m, 7) = \frac{3}{7}$. PROVE these equalities (this will involve showing both $f(m, 7) \leq \frac{3}{7}$ and $f(m, 7) \geq \frac{3}{7}$.

SOLUTION

Omitted