

### **Homework 05, Morally due Mon Mar 22, 9:00AM**

For Programming Problems: Send your code to Emily by email. Send the actual .java/.py/ect file. You need to use your .umd email address or it will not send. In your pdf, you must have the output your code provides. You can screenshot this or type it in. Hint: Use Python.

1. (0 points but if you miss the midterm that means you got this wrong retroactively and you will lose a lot of points). When is the SECOND midterm? By what day do you need to tell Dr. Gasarch that you cannot make the midterm (if you cannot and know ahead of time)?
2. (10 points) Write an expression with quantifiers that means there are NO elements between  $x$  and  $y$ . Call it NOBET for later problems.

**SOLUTION**

$$\text{NOBET}(x, y) \equiv (\forall z)[z \leq x \vee z \geq y]$$

**END OF SOLUTION**

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3. (15 points) For each sentence below give

(1) a domain  $D \subseteq \mathbb{R}$  such that over that domain the statement is TRUE, and

(2) a domain  $D \subseteq \mathbb{R}$  such that over that domain the statement is FALSE.

Explain why your answer works.

(a) (5 points)  $(\forall x)(\exists y)[x < y \wedge \text{NOBET}(x, y)]$

**SOLUTION**

TRUE when  $D = \mathbb{Z}$ . If  $x \in \mathbb{Z}$  then let  $y = x + 1$ .

FALSE when  $D = \mathbb{Q}$ . If  $x = 0$  then for any  $y > 0$  you have  $\neg \text{NOBET}(0, y)$  by taking  $\frac{y}{2}$ .

**END OF SOLUTION**

(b) (10 points)  $(\exists x)[(\forall y \neq x)[y < x \wedge (\exists z)[y < z \wedge \text{NOBET}(y, z)]]]$

**SOLUTION**

TRUE when  $D = \{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots\} \cup \{1\}$ .

$x = 1$ . For all  $y \neq 1$ , clearly  $y < x = 1$ . If  $y = \frac{2^{n-1}}{2^n}$  then take  $z = \frac{2^n}{2^{n+1}}$ .

FALSE when  $D = \mathbb{Q}$ . There is no max element.

**END OF SOLUTION**

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4. (15 points)

- (a) (0 points) Write a program that will, given a number, determine the LEAST number of SQUARES that add up to it and output a way to do that. For example:

On input 6 it outputs

$$6 = 2^2 + 1^2 + 1^2, \text{ so } 3.$$

- (b) (10 points) Run your program on  $1, \dots, 1000$ . Give both ALL of your data and also the numbers that needed 4 squares.

- (c) (5 points) Make a conjecture of the form:

If  $x$  satisfies property BLAH then  $x$  requires 4 squares.

Every number between 1 and 1000 that has property BLAH has to require 4 squares (though we do not insist on the converse).

There must be an infinite number of numbers that satisfy BLAH.

(EXAMPLE which is not true: BLAH is that  $x$  is the product of a prime and a square.)

**SOLUTION OMITTED**

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5. (15 points)

- (a) (0 points) Write a program that will, given a number, determine the LEAST number of CUBES that add up to it and output a way to do that. For example:

On input 15 it outputs

$$15 = 2^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 \text{ so } 8.$$

- (b) (10 points) Run your program on  $1, \dots, 1000$ . Let MAX be the MAX number of cubes that a number needed.

For all  $1 \leq i \leq MAX$  state how many numbers needed  $i$  cubes.

- (c) (5 points) Make a conjecture of the form:

For all sufficiently large  $n$ ,  $n$  can be written as the sum of X cubes.

**SOLUTION**

OMITTED

**END OF SOLUTION**

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6. (15 points) We are working in binary. When we input 3 bits to a circuit we think of it as a NUMBER in base 2.

000=0

001=1

010=2

011=3

100=4

101=5

110=6

111=7

The output will be 1 bits.

- (a) (10 points) Write a truth table with 3 inputs and 1 outputs for the following function:

$$f(xyz) = \begin{cases} 1 & \text{if } xyz \text{ is prime;} \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

(0 and 1 are not primes.)

So for example

$f(111) = 1$  since 7 is prime.

- (b) (5 points) Using the truth table write a formula for the function  $f$ . DO NOT SIMPLIFY as it will make it harder for Emily to grade.
- (c) (0 points and don't turn in) Draw a circuit for the function  $f$ .

### SOLUTION

a)

$x$	$y$	$z$	$f(xyz)$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

b) OMITTED.

**END OF SOLUTION**

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7. (15 points)

- (a) (5 points) Show that there exists a Boolean formula on 10 variables such that IF you did the truth table there would be exactly 239 rows that return TRUE.
- (b) (10 points) For which  $k, n$  does there exist a boolean Formula on  $n$  variables such that IF you did the truth table there would be exactly  $k$  rows that return TRUE.

**SOLUTION**

a) First write down a truth table with the output row blank for a 10-variable boolean formula. There will be 1024 rows. Now make the first 239 of the rows T and the rest F.

If you then covert this Truth Table into a boolean formula, you get a formula on 10 variables where the truth table has exactly 239 rows true.

b) The answer is  $(k, n)$  such that  $k \leq 2^n$ .

First write down a truth table with the output row blank for a  $n$ -variable boolean formula. There will be  $2^n$  rows. Now make the first  $k$  of the rows T and the rest F.

If you then covert this Truth Table into a boolean formula, you get a formula on  $n$  variables where the truth table has exactly  $k$  rows true.

**END OF SOLUTION**

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8. (15 points) Let  $\phi(x_1, \dots, x_n)$  and  $\psi(x_1, \dots, x_n)$  be Boolean formulas on  $n$  variables. They are *equivalent* if they have the same truth table.

We want to count the number of boolean formula but NOT want to count  $\phi$  and  $\psi$  if they are equivalent.

How big is the largest set of non-equivalent Boolean Formulas on  $n$  variables.

**SOLUTION**

We only have to look at the number of truth tables. There are  $2^n$  rows and each row has either T or F. So there are  $2^{2^n}$  non-equivalent Boolean Formulas.

**END OF SOLUTION**