Homework 06, Morally due Mon Mar 29, 9:00AM

For Programming Problems: Send your code to Emily by email. Send the actual .java/.py/ect file. You need to use your .umd email address or it will not send. In your pdf, you must have the output your code provides. You can screenshot this or type it in. Hint: Use Python.

1. (0 points but if you miss the second midterm that means you got this wrong retroactively and you will lose a lot of points). When is the SECOND midterm? By what day do you need to tell Dr. Gasarch that you cannot make the midterm (if you cannot and know ahead of time)?

   HINT: The date of the second midterm has been CHANGED. It is now Thursday April 8, 2021. (We did this so that we could have a HW due and gone over BEFORE it.)

GOTO NEXT PAGE

2. (0 points but you will need this for the next problem.)

   Let ONEFOUR be the set

$$\{n : n \equiv 1 \pmod 4\}.$$

   Write a program that will, given a natural number $n \ge 1$ do the following.

   For $i = 1$ to n

   (a) $A[i] = 4i + 1$.
   (b) $B[i]$ is the factorization of $A[i]$ in ONEFOUR.
   (c) Count how many elements of $A[1], \ldots, A[n]$ are PRIME, are the product of 2 primes, are the product of 3 primes, etc. Store that list in $C[i]$.

   Example:

   $A[1] = 5$, $B[1] = (5)$, $C[1] = (1)$.

   $A[2] = 9$, $B[2] = (9)$, $C[2] = (2)$.

   $A[3] = 13$, $B[3] = (13)$, $C(3) = (3)$.

   $A[4] = 17$, $B[4] = (17)$, $C(4) = (4)$.

   $A[5] = 21$, $B[5] = (21)$, $C(5) = (5)$.

   $A[6] = 25$, $B[6] = (5, 5)$, $C(6) = (5, 1)$.

GOTO NEXT PAGE

3. (20 points) Graph the following functions and try to see what they look like.

   (a) $f_1(i)$ is the number of primes in ONEFOUR of the first i elements of ONEFOUR (so the first coordinate of $C[i]$).

   (b) $f_2(i)$ is the number of products of 2 primes in ONEFOUR of the first i elements of ONEFOUR (so the second coordinate of $C[i]$.

   (c) Etc.

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4. (20 points) Complete this sentence and prove it:

If $n_1 \equiv 5 \pmod{10}$ *and* $n_2 \equiv 10 \pmod{20}$ *then* $n_1 n_2 \equiv X \pmod{Y}$

SOLUTION

We show how to think about the problem and then give the answer.

$n_1 \equiv 5 \pmod{10}$, so $n_1 = 10k_1 + 5$ for some k_1.

$n_2 \equiv 10 \pmod{20}$, so $n_2 = 20k_2 + 10$ for some k_2.

SO

$n_1 n_2 = 200k_1k_2 + 100k_1 + 100k_2 + 50$

AH, so

$n_1 n_2 = 100(2k_1k_2 + k_1 + k_2) + 50$

SO

$n_1 n_2 \equiv 50 \pmod{100}$.

END OF SOLUTION

5. (20 points) For this problem all numbers mentioned are in R unless otherwise restricted. So, for example,

$x \notin \mathsf{Q}$ means $x \in \mathsf{R} - \mathsf{Q}$.

For this problem you may assume $\pi \notin \mathsf{Q}$.

(a) (7 points) Prove or disprove:
For all $x, y \in \mathsf{Q} - \{0\}$, $x\pi + y \notin \mathsf{Q}$.
SOLUTION
Let $x, y \in \mathsf{Q} - \{0\}$.
Assume, by way of contradiction, that $x\pi + y \in \mathsf{Q}$.
so there exists $a, b \in \mathsf{N}$ such that
$x\pi + y = \frac{a}{b}$.
Hence
$\pi = (\frac{a}{b} - y)/x$.
Since rationals are closed under $+, -, \div, \times$ we have that π is rational, which is a contradiction.
END OF SOLUTION

(b) (7 points) Prove or Disprove:

$$(\forall x, z \in \mathsf{Q})[x < z \rightarrow (\exists y \notin \mathsf{Q})[x < y < z]$$

SOLUTION TRUE
Let $x, z \in \mathsf{Q}$ with $x < z$. Let $n \in \mathsf{N}$ be such that $\frac{\pi}{n} < z - x$.
Then we have

$$x < x + \frac{\pi}{n} < z$$

By Part 1 $x + \frac{\pi}{n} \notin \mathsf{Q}$.
END OF SOLUTION

(c) (6 points) Prove or Disprove:

$$(\forall x, z \notin \mathsf{Q})(\exists y \in \mathsf{Q})[x < y < z]$$

SOLUTION
TRUE

We will assume $x, z \in (0,1)$. We leave it to the reader to adjust the proof for other cases.

Let $x = 0.x_1 x_2 \cdots$.

Let $z = 0.z_1 z_2 \cdots$.

Since $x < z$ there exists a least i such that:

$x_1 = z_1$

$x_2 = z_2$

$\vdots$

$x_{i-1} = z_{i-1}$

$x_i < z_i$.

Let

$y = x_1 \cdots x_{i-1} z_i$.

This is a finite expansion so $y \in \mathsf{Q}$.

Clearly

$x < y < z$.

END OF SOLUTION

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6. (20 points) Bill tells Alice and Bob to do the following:

   If you're happy and you know it clap your hands

   (If you Google that phrase on You Tube you will hear a very annoying song.)

   Alice and Bob follow Bill's command.

   - Alice is clapping her hands.
   - Bob is NOT clapping his hands.

   What you can conclude about Alice? About Bob?

   SOLUTION

   Lets first write the command in logic.

   H is happy, K is knowing you are happy, and C is clap.

   So Bill is saying:

   $H \wedge K \rightarrow C$

   Alice is clapping her hands. From this you can conlude NOTHING. She may be clapping for … who-knows-why!

   Bob is NOT clapping. Hence EITHER

   Bob is NOT happy

   or

   Bob IS HAPPY but does not know it (weird!).

   or

   Bob knows he is happy but he is not (even weirder).

   (If you missed that last case claiming that K can only happen if H happens then you will NOT lose points.)

   END OF SOLUTION

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7. (20 points) Let $x_1, x_2, y_1, y_2, m \in \mathsf{N}$. $m \geq 2$. ALL $\equiv$ are mod m. Assume $x_1 \equiv x_2$ and $y_1 \equiv y_2$. (Hence

$x_1 = x_2 + k_x m$ for some $k_x \in \mathsf{Z}$

and

$y_1 = y_2 + k_y m$ for some $k_y \in \mathsf{Z}$.

)

(a) (7 points) Prove or disprove: $x_1 + y_1 \equiv x_2 + y_2$.

SOLUTION

TRUE

$x_1 = x_2 + km_x$

$y_1 = y_2 + km_y$

ADD these together to get:

$$x_1 + y_1 = x_2 + y_2 + k(m_x + m_y).$$

$x_1 + y_1 \equiv x_2 + y_2$.

END SOLUTION

(b) (7 points) Prove or disprove: $x_1 y_1 \equiv x_2 y_2$.

SOLUTION

TRUE

$x_1 = x_2 + km_x$

$y_1 = y_2 + km_y$

MULTIPLY these together to get:

$$x_1 y_1 = x_2 y_2 + km_x y_1 + km_y x_2 + k^2 m_x m_y = x_2 y_2 + k(m_x y_1 + m_y x_2 + km_x m_y)$$

So

$x_1 y_1 \equiv x_2 y_2$.

END SOLUTION

(c) (7 points) Prove or disprove: $x_1^{y_1} \equiv x_2^{y_2}$.

SOLUTION

FALSE.

We NEED a counterexample to show that this is FALSE.

$m = 5$.

$x_1 = 2$, $x_2 = 2$, $y_1 = 7$, $y_2 = 8$.

$x_1^{x_2} = 2^2 = 4 \equiv 4 \pmod{5}$.

$y_1^{y_2} = 7^8 = 5764801 \equiv 1 \pmod{5}$.

AH- $1 \not\equiv 4 \pmod{5}$.

END SOLUTION