Homework 07, Morally due Mon Apr 5, 9:00AM

For Programming Problems: Send your code to Emily by email. Send the actual .java/.py/ect file. You need to use your .umd email address or it will not send. In your pdf, you must have the output your code provides. You can screenshot this or type it in. Hint: Use Python.

1. (0 points but if you miss the second midterm that means you got this wrong retroactively and you will lose a lot of points). When is the SECOND midterm? By what day do you need to tell Dr. Gasarch that you cannot make the midterm (if you cannot and know ahead of time)?

   HINT: The date of the second midterm has been CHANGED. It is now Thursday April 8, 2021. (We did this so that we could have a HW due and gone over BEFORE it.)

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2. (30 points) In the country of Fredonia they use two kinds of coins: 6-cent and 7-cent.

   (a) (10 points) Find the number n_0 such that

      - $(\forall n \ge n_0)(\exists n_1, n_2 \in \mathsf{N})[n = 6n_1 + 7n_2]$, and
      - $\neg(\exists n_1, n_2)[n_0 - 1 = 6a + 7b]$.

   (b) (20 points) Prove both parts.

   HINT Get some empirical evidence, either by hand or by program. You do not have to send Emily your program if you made one.

SOLUTION

We first compute which numbers can and cannot be sums of 6's and 7's.

You CANNOT get 1,2,3,4,5. (AH- 5 of them)

You CAN get 6,7.

You CANNOT get 8,9,10,11. (AH- 4 of them. Trend?)

You CAN get 12,13,14.

You CANNOT get 15,16,17. (AH- 3 of them. YES, Trend!)

You CAN get 18,19,20,21.

You CANNOT get 22,23. (AH-2 of them!)

You CAN get 24,25,26,27,28.

You CANNOT get 29. (AH- only 1. This could be the last one.)

I'll guess that $n_0 = 30$.

Theorem There is NO way to write 29 as 6's and 7's.

Proof Assume, BWOC, that

$29 = 6n_1 + 7n_2$.

MOD out by 7

$1 \equiv 6n_1 \pmod{7}$.

Trial and error shows that $n_1 \equiv 6 \pmod{7}$. So $n_1 \geq 6$.

But then $6 \times n_1 \geq 36 > 29$.

So CANNOT have $29 = 6n_1 + 7n_2$.

END OF PROOF

Theorem $(\forall n \ge 30)(\exists n_1, n_2)[30 = 6n_1 + 7n_2]$.

Proof

Base Case $n = 30 = 6 \times 5$ so $n_1 = 5$, $n_2 = 0$.

IH $n \ge 30$ and there exists n_1, n_2, $n = 6n_1 + 7n_2$.

IS Intuition: How can we swap coins in and out and get 1 more cent?

Obvious one: can swap out a 6-cent coin and swap in a 7-cent coin. Need to HAVE a 6-cent coin.

Case 1 $n_1 \ge 1$.

$n = 6n_1 + 7n_2$

$n + 1 = 6(n_1 - 1) + 7(n_2 + 1)$.

(Needed $n_1 \ge 1$ so that $n_1 - 1 \ge 0$.)

Intuition: Want a way o swap out some number of 7-cent coins and swap in some 6-cent coins. Swap out five 7-cent coins and swap in six 6-cent coins adds 1.

Case 2 $n_2 \ge 5$.

$n = 6n_1 + 7n_2$

$n + 1 = 6(n_1 + 6) + 7(n_2 - 5)$.

(Needed $n_2 \ge 5$ so that $n_2 - 5 \ge 0$.)

Intuition: Now need to show that either Case 1 or Case 2 occurs.

Case 3 $n_1 \le 0$ and $n_2 \le 4$.

Then

$n = 6n_1 + 7n_2 \le 6 \times 0 + 7 \times 4 = 28$.

BUT we were told $n \ge 30$. So this case cannot occur.

END OF PROOF

END OF SOLUTION

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3. (30 points) Let P be some statement about INTEGERS.

   Dr. Gasarch proved $P(-8)$.

   Dr. Gasarch then proved $(\forall n \in \mathsf{Z})[P(n) \rightarrow P(n+3)]$.

   Fill in XXX in the following sentence

   $P(n)$ holds iff n satisfies XXX.

   SOLUTION

   P hold for -8.

   $P(-8) \rightarrow P(-5)$, so we know $P(-5)$ holds.

   $P(-5) \rightarrow P(-2)$, so we know $P(-2)$ holds.

   $P(-2) \rightarrow P(-2)$, so we know $P(1)$ holds.

   So so far the set is

$$\{-8, -5, -2, 1, \ldots\}$$

   Lets write that in terms of mods

$$\{n : n \geq -8 \wedge n \equiv 1 \pmod 3\}.$$

   SO

   $P(n)$ holds iff $((n \geq -8) \wedge (n \equiv 1 \pmod 3))$.

   END OF SOLUTION

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4. (40 points) Let a_n be defined as follows

   $a_0 = 17$

   $(\forall n \geq 1)[a_n = a_{\lfloor n/2 \rfloor} + \lfloor \sqrt{n} \rfloor]$

   (a) Write a program that computes a_n and graph it.

   (b) Use your graph to GUESS a number $A, B \in \mathsf{N}$ such that $a_n \leq A\sqrt{n} + B$.

   (c) Use constructive induction to DERIVE A, B such that $a_n \leq A\sqrt{n} + B$.

   (You allowed to *cheat* and always assume $n/2$ and $\sqrt{n}$ are natural numbers, go avoid dealing with floors.)

   (This does not have to agree with the answer you got in the last part.)

SOLUTION

Base Case $a_0 = 17$ so we need $1 \leq A\sqrt{0} + B$ so $B \geq 17$.

IH $(\forall n' < n)[a_{n'} \leq A\sqrt{n'} + B]$.

IS

$$a_n = a_{n/2} + \sqrt{n} \leq (A\sqrt{n/2} + B) + \sqrt{n}$$

We will pick A, B so that this is $\leq A\sqrt{n} + B$.

$$A\sqrt{n/2} + B + \sqrt{n} \leq A\sqrt{n} + B$$

$$A\sqrt{n/2} + \sqrt{n} \leq A\sqrt{n}$$

$$A\frac{1}{\sqrt{2}} + 1 \leq A$$

$$1 \leq A - A\left(\frac{1}{\sqrt{2}}\right) = A\left(1 - \frac{1}{\sqrt{2}}\right) = A\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

So we can take

$$A = \frac{\sqrt{2}}{\sqrt{2}-1} = 2 + \sqrt{2}$$

END OF SOLUTION