250 MIDTERM I

Read these instructions:

1. This is a open book, open notes, open web exam.
2. There are 4 problems which add up to 75 points. (Recall that the Midterm Muffin Problem is worth 15 points, and the Midterm Grid Coloring Problem is 10 points, and they were both already handed in.) The exam is 2 hours.
3. For each question show all of your work and **write legibly**. **Clearly indicate** your answers. No credit for illegible answers.
4. Please write out the following statement: "*I pledge on my honor that I will not give or receive any unauthorized assistance on this examination.*"

5. Fill in the following:

NAME :
SIGNATURE :
SID :
SECTION NUMBER :

1. (20 points) In Fredonia everyone's first name either begins with an A a B or a C. People whose names begin with an A are called A-Types. People whose names begin with a B are called B-Types. People whose names begin with a C are called C-Types.

   (a) The Fredonia Senate has x A-types, y B-types and z C-types where $x, y, z \geq 10$.
   Let $1 \leq x' \leq x$ and $1 \leq y' \leq y$ and $1 \leq z' \leq z$. The President will choose a committee of x' A-types, y' B-types, and z' C-types.
   How many ways can the President form a committee?

   (b) The President wants exactly one of Alice, Bob, Carol to be on the committee. NOW how many ways can the President form a committee?

SOLUTION TO PROBLEM ONE

a) $\binom{x}{x'}\binom{y}{y'}\binom{z}{z'}$

b) There are three disjoint cases

Alice is on the subcommittee but neither Bob nor Carol are.
$\binom{x-1}{x'-1}\binom{y-1}{y'}\binom{z-1}{z'}$
Bob is on the subcommittee but neither Alice nor Carol are.
$\binom{x-1}{x'}\binom{y-1}{y'-1}\binom{z-1}{z'}$
Carol is on the subcommittee but neither Alice nor Bob are.
$\binom{x-1}{x'}\binom{y-1}{y'}\binom{z-1}{z'-1}$
So the answer is:

$$\binom{x-1}{x'-1}\binom{y-1}{y'}\binom{z-1}{z'}+\binom{x-1}{x'}\binom{y-1}{y'-1}\binom{z-1}{z'}+\binom{x-1}{x'}\binom{y-1}{y'}\binom{z-1}{z'-1}$$

GRADING Some students had x instead of $x-1$ in the top part of $\binom{x}{x'-1}$, or similar for some of the other summands. This was only 5 points off since Bill's old midterm had the same mistake. However, if we put this on a later exam then this mistake will get you a 0 since NOW the points has been clarified.

END OF SOLUTION

2. (20 points) In this problem if I have a die then p_i or q_i is the prob that the die comes up i. The die may be unfair, so could have $p_i \neq p_j$.

   (a) Bill has a 2-sided die and a 3-sided die.
   The 2-sided die has $p_1 = p_2 = \frac{1}{2}$.
   The 3-sided die has $q_1 = \frac{1}{2}$, $q_2 = 0$, $q_3 = \frac{1}{2}$.
   Bill rolls the 2-sided die and the 3-sided die, and computes the sum.

      i. What is the probability that the sum is 2?
      ii. What is the probability that the sum is 3?
      iii. What is the probability that the sum is 4?
      iv. What is the probability that the sum is 5?

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(b) After Bill rolls the die and computes the sum, Emily also computes the sum. But then Emily does the following:
With probability $\frac{1}{3}$ she adds one to the total.
With probability $\frac{2}{3}$ she subtracts one from the total.
CLARIFICATION: Emily just does ONE thing: She will either add one or subract 1.

i. What is the probability that the final number is 1?
ii. What is the probability that the final number is 2?
iii. What is the probability that the final number is 3?
iv. What is the probability that the final number is 4?
v. What is the probability that the final number is 5?
vi. What is the probability that the final number is 6?

SOLUTION

FIRST PART: BILL ROLLS THE DICE!

(a) What is the probability that the sum is 2?
$p_1q_1 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

(b) What is the probability that the sum is 3?
$p_1q_2 + q_1p_2 = 0 + \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

(c) What is the probability that the sum is 4?
$p_1q_3 + p_2q_2 = \frac{1}{2} \times \frac{1}{2} + 0 = \frac{1}{4}$.

(d) What is the probability that the sum is 5?
$p_2q_3 = \frac{1}{2} \times \frac{1}{2} + 0 = \frac{1}{4}$.

SECOND PART: EMILY ADDS OR SUBTRACTS FROM THE SUM:

(a) What is the probability that the final number is 1?
This only happens if the original sum is 2 and we subtract 1. So $\frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$.

(b) What is the probability that the final number is 2?
This only happens if the original sum is 3 and we subtract 1.
$\frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$.

(c) What is the probability that the final number is 3?

This happens if the original sum is 2 and we ADD ONE or the original sum is 4 and we SUBTRACT 1, so
$\frac{1}{4} \times \frac{2}{3} + \frac{1}{4} \times \frac{1}{3} = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$.

(d) What is the probability that the final number is 4?

This happens if the original sum is 3 and we ADD ONE or the original sum is 5 and we SUBTRACT 1, so identical to the above so $\frac{1}{4}$.

(e) What is the probability that the final number is 5?

This happens if the original sum is 4 and we ADD ONE
$\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$.

(f) What is the probability that the final number is 6?

This happens if the original sum is 5 and we ADD ONE
$\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$.

GRADING Some students interpreted the problem as Emily does TWO things. If they did this then we graded it as if it was that problem.

3. (20 points) Use a combinatorial argument to show

$$3^n = \sum_{a+b+c=n} \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}$$

SOLUTION

Consider the question: How many ways can you 3-color $\{1, \ldots, n\}$?

One answer is, for each number in $\{1, \ldots, n\}$ color it RED or WHITE or BLUE. Thats 3^n.

Another answer is that every coloring has a RED, b WRITE, and c BLUE for some $a + b + c = n$. If there are a RED, b WHITE, c BLUE then there are

$$\binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}$$

ways to color $\{1, \ldots, n\}$.

GRADING If a student did an Algebraic proof and it was correct they got 5 points. If a student's answer was nonsense they got 0. (If you do a regrade request on an answer we thought was nonsense then you may lose points if your explanation is nonsense. *Warning:* Past experience tells us that this is likely.)

4. (15 points) We are back in Fredonia!

(a) (7 points) In Fredonia they have two types of coins: 5-cent coins and 8-cent coins. President Mae has just proven that, for LARGE ENOUGH n, one can always form n cents with some 5-cent coins and some 8-cent coins (note that 1,2,3,4,5,6,7 cannot be formed, plus some other amounts, for example 9). EXPRESS Mae's theorem using quantifiers which range over N.

SOLUTION

$$(\exists a)(\forall b)[b \ge a \rightarrow (\exists c, d)[b = 5c + 8d]].$$

END OF SOLUTION

(b) (8 points) A new government comes in and wants to change the monetary system. They want to use x-cent coins and y-cent coins and are looking for good values of (x, y). What is good? They still want that for LARGE ENOUGH n, one can always form n cents. EXPRESS the property they want (x, y) to have using quantifiers which range over N. So we want an expression

$GOOD(x, y)$ which uses quantifiers. Note that x, y will NOT be quantified over.

SOLUTION

$$(\exists a)(\forall b)[b \ge a \rightarrow (\exists c, d)[b = xc + yd].$$

END OF SOLUTION

Grading This applies to both parts. If a student did not take into account the 'for large enough' but everything else is correct, they got 10 points. If a students answer was mostly nonsense they got 5 points. (If you do a regrade request on an answer we thought was mostly nonsense then you may lose points if your explanation is nonsense. *Warning:* Past experience tells us that this is likely.)