Take-Home Midterm Two Problems,
Morally due Mon April 5, 9:00AM, Dead Cat-Th April 8 2:00PM

On Thursday April 8 you will be taking your second midterm. It will be 2-hours and be 70 points. Why 70 points? Because THIS document has THREE problems that you will do ahead of time and have plenty of time to do, which is worth 30 points.

This must be handed in typed and easy to read.

Here are the problems:

1. (10 points) Let L_1 be the ordering $\mathsf{Z}+\mathsf{Q}$, which is a copy of Z followed by a copy of Q.

   Let L_2 be the ordering $\mathsf{Q}+\mathsf{Z}$, which is a copy of Q followed by a copy of Z.

   (a) (5 points) Give a sentence with quantifiers which is TRUE in L_1 but FALSE in L_2.

   (b) **SOLUTION**

   We note that if w is in the Q-part of $\mathsf{Z}+\mathsf{Q}$ then EVERY pair after w has density. But there is NO element of L_2 with this property. Formally the statement is

   $$(\exists w)(\forall x)(\forall y)[w < x < y \rightarrow (\exists z)[x < z < y]$$

   END OF SOLUTION

   (c) (0 points) What is the quantifier depth of your sentence? Let it be k_1.

   SOLUTION

   DELAYED for same reason as problem 1a.

   END OF SOLUTION

   (d) (5 points) Give a value k such that SPOILER wins (L_1, L_2, k). Describe SPOILER's strategy taking into account ANYTHING that DUP does.

   SOLUTION

   $L_1 = \mathsf{Z}+\mathsf{Q}$.

   $L_2 = \mathsf{Q}+\mathsf{Z}$.

Intuition We note that if w is in the Q-part of $\mathsf{Z}+\mathsf{Q}$ then EVERY pair after w has density. But there is NO element of L_2 with this property. We can USE this property.

Convention If SP plays in the Q part of L_1 then DUP will play in the Q-part of L_2 unless it leads to an immediate lose. All other combos as well.

Convention We will call the points in L_1 x and the points in L_2 x'. (others as well: y, y', etc.)

Strategy

i. SP picks w in the Q part of L_1.

ii. DUP is forced to pick w' in the Q part of L_2.

iii. SP picks x' in the Z part of L_2. Note that $w' < x'$ Hence DUP must pick a x such that $w < x$. Since w is in the Q part of L_1, x is in the Q part of L_1.

iv. SP picks $y' = x' + 1$ in L_2.

v. DUP is forced to pick some $y > x$ in the Q part of L_1.

vi. SP picks z between x and y. DUP cannot respond.

END OF SOLUTION

(e) (0 points) Is $k_1 = k_2$

SOLUTION

DELAYED FOR SAME REASON.

END OF SOLUTION

2. (10 points) Give TWO DIFFERENT proofs of the following:

$$(x_1, \ldots, x_n \in \mathsf{R}^+)\left[\frac{\sum_{i=1}^n x_i}{n} \geq (\prod_{i=1}^n x_i)^{1/n}\right].$$

This is called the Arithmetic-Geometric Inequality (or similar names).

Strong Advice Look up proofs of this on the web and put them in your own words.

(If I prove this in class you may use that as one of your proofs. Put it in your own words.)

SOLUTION

Solution omitted.

END OF SOLUTION

3. (10 points) The proofs that $\sqrt{n}$ is irrational are all the same! Write a program that will, given $n \in \mathsf{N}$, produce the PROOF that $\sqrt{n}$ is irrational OR tell you where the attempt fails.

SOLUTION

Solution omitted.

END OF SOLUTION