**Combinatorial Proof of Midterm Problem 3**

**Using Trinomial Theorem**

## Problem 3

Use a combinatorial argument to show $3^n = \sum_{a+b+c=n} \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}$

## Solution Using Trinomial Theorem

The trinomial theorem is

$$(x+y+z)^n = \sum_{a+b+c=n, 0 \le a,b,c \le n} \frac{n!}{a!b!c!} x^a y^b z^z$$

If you plug in $x = y = z = 1$ you get $3^n = \sum_{a+b+c=n, 0 \le a,b,c \le n} \frac{n!}{a!b!c!}$

So we have to prove that

$$\sum_{a+b+c=n, 0 \le a,b,c \le n} \frac{n!}{a!b!c!} = \sum_{a+b+c=n, 0 \le a,b,c \le n} \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}.$$

So we need to prove

$$(\forall a,b,c,n \in \mathsf{N})[a+b+c=n \to \frac{n!}{a!b!c!} = \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}.$$

This we can prove combinatorially.

Fix $a, b, c$ such that $0 \le a, b, c \le n$ and $a + b + c = n$.

The LHS is the number of ways to permute

$a$ $A$'s, $b$ $B$'s, and $c$ $C$'s.

The RHS solves the same problem a diff way: First CHOOSE which $a$ spots to put the $A$'s. Then there are $n - a$ spotes left, chose which $b$ spots to put the $B$'s. Then there are $n - a - b$ spots left, choose whihc $c$ spots to put the $c$'s. (The last one is really just 1 since $n - a - b = c$.)