250 MIDTERM II

Do not open this exam until you are told. Read these instructions:

1. This is a closed book exam, though ONE sheet of notes is allowed. **No calculators, or other aids are allowed**. If you have a question during the exam, please raise your hand.
2. There are 4 problems which add up to 100 points. The exam is 1 hours 15 minutes. (You shouldn't need that much.)
3. For each question show all of your work and **write legibly**. **Clearly indicate** your answers. No credit for illegible answers.
4. Please write out the following statement: "*I pledge on my honor that I will not give or receive any unauthorized assistance on this examination.*"

5. Fill in the following:

NAME :
SIGNATURE :
SID :
SECTION NUMBER :

1. (25 points)

   (a) Let $x, y \geq 10$. There are x males and y females on the committee to revise CMSC 250. Let $1 \leq x' \leq x$ and $1 \leq y' \leq y$. The dean will choose a subcommittee of x' males and y' females. How many ways can the Dean do this?

   (b) The Dean does not want Alice (a female) and Bob (a male) to both be on the subcommittee. NOW how many ways can the Dean choose the subcommittee.

SOLUTION TO PROBLEM ONE

a) $\binom{x}{x'}\binom{y}{y'}$

b) There are three disjoint cases

Alice is on the subcommittee but Bob is not. Of the x males you DO NOT pick Bob, but you still need x', so thats $\binom{x-1}{x;}$. Of the y females you DO want Alice, but you still need $y'-1$ out of the remaining $y'-1$, so thats $\binom{y-1}{y'-1}$. Hence we have:

$\binom{x-1}{x'}\binom{y-1}{y'-1}$

Bob is on the subcommittee but Alice is not. Of the x males you DO pick Bob, you still need $x'-1$ more males. Thats $\binom{x-1}{x'-1}$. Of the y females you DO NOT pick Alice, you still need y' females, so thats $\binom{y-1}{y'}$. Hence we have:

$\binom{x-1}{x'-1}\binom{y-1}{y'}$

Neither is on the subcommittee:

$\binom{x-1}{x'-1}\binom{y-1}{y'-1}$

So the answer is

$$\binom{x-1}{x'}\binom{y-1}{y'-1} + \binom{x-1}{x'-1}\binom{y}{y'} + \binom{x-1}{x'-1}\binom{y-1}{y'-1}$$

2. (25 points) What is the coefficient of $x^{10}y^5$ in

$$(x+2y)^{15}$$

SOLUTION TO PROBLEM TWO

The number of terms that have 10 x's and 5 y's is $\binom{15}{10}$. But every time you get a y you also get a 2, so its

$$\binom{15}{10}2^5$$

3. (25 points) Let $k, n \in \mathsf{N}$ with $3 \le k \le n$. Fill in the blanks in the following statement. Describe your reasoning. BLANK will be a function of k, n, for example BLANK could be $k + \lceil \lg n \rceil$ (it is NOT that!).

   If $A \subseteq \{1, \ldots, n\}$ and $|A| = k$ then at least BLANK subsets of A OF SIZE 3 have the same SUM.

   (NOTE the OF SIZE 3)

   Make BLANK as large as possible using the methods of this course.

   SOLUTION TO QUESTION THREE

   There are $\binom{k}{3}$ subsets of A OF SIZE 3.

   The min sum is $1 + 2 + 3 = 6$

   The max sum is $n + (n-1) + (n-2) = 3n - 3$

   Hence the total number of sums is $3n - 2$.

   Hence the number of sets that have the same sum is at least

$$\left\lceil \frac{\binom{k}{3}}{3n-2} \right\rceil .$$

4. **NOT RELEVANT TO MIDTERM ONE FOR SPRING 2021**
(25 points) Let $T(n)$ be defined by $T(1) = 0$

$$(\forall n \geq 1)\left[T(n) = T\left(\left\lfloor \frac{n}{11} \right\rfloor\right) + T\left(\left\lfloor \frac{2n}{11} \right\rfloor\right) + T\left(\left\lfloor \frac{3n}{11} \right\rfloor\right) + 2n\right]$$

Use constructive induction to find a constant $A \in \mathsf{N}$ such that

$$(\forall n \geq 0)[T(n) \leq An].$$

NOTE- if you do not have enough room go to the NEXT page.

ONLY USE THIS FOR PROBLEM FOUR

IF YOU TEAR THIS PAGE OUT YOU WILL LOSE 10 POINTS

SOLUTION TO PROBLEM FOUR

We do a proof that $T(n) \leq An$ and see what conditions on A we get.

Base Case: $T(1) = 0 \leq A \times 1$. No condition needed here.

IH: For all $n' < n$ $T(n') \leq An'$

IS:

By definition:

$$T(n) = T\left(\left\lfloor \frac{n}{11} \right\rfloor\right) + T\left(\left\lfloor \frac{2n}{11} \right\rfloor\right) + T\left(\left\lfloor \frac{3n}{11} \right\rfloor\right) + 2n]$$

By the IH:

$$\leq \frac{An}{11} + \frac{2An}{11} + \frac{3An}{11} + 2n$$

WANT:

$$\frac{An}{11} + \frac{2An}{11} + \frac{3An}{11} + 2n \leq An$$

$$\frac{A}{11} + \frac{2A}{11} + \frac{3A}{11} + 2 \leq A$$

$$\frac{6A}{11} + 2 \leq A$$

$$2 \leq \frac{5A}{11}$$

$$2 \times \frac{11}{5} \leq A$$

$$A = \frac{22}{5}$$

Need

$$A\alpha n + A\beta n + \gamma n \leq An$$

$$A\alpha + A\beta + \gamma \leq A$$

$$\gamma \leq A(1 - \alpha - \beta)$$

$$A \geq \frac{\gamma}{1 - \alpha - \beta}$$

So take $A = \frac{\gamma}{1-\alpha-\beta}$.