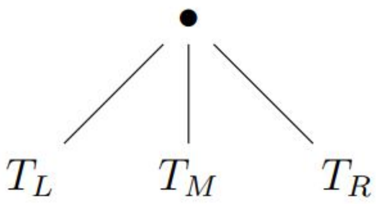
Tree Induction

250H

Suppose we define a three-tree recursively as follows:

- A single node • is a three-tree
- If T_L, T_M, T_R are three-trees, then



is a three-tree

Denote $N(T)$ = the number of nodes in the three-tree T. Define $h(T)$ = the height of the three-tree T recursively as:

- 0 if T = a single node
- $1 + \max\{T_L, T_M, T_R\}$ if T = a node with three children T_L, T_M, T_R

Prove that $N(T) \leq \frac{3^{h(T)+1}-1}{2}$ for all three-trees T.

Proof by induction on n, the stage in which T was formed.

Baes Case: Consider a single node. Then, $N(T) = 1$ and $h(T) = 0$ so

$$1 \leq \frac{3^{0+1} - 1}{2}$$

.

$$1 \leq 1$$

Inductive Hypothesis: For some T, $N(T) \leq \frac{3^{h(T)+1}-1}{2}$.

Inductive Step: Let T' be a three tree with a root and sub trees T'_L, T'_M, T'_R.
$N(T') = 1 + N(T'_L) + N(T'_M) + N(T'_R)$.
$h(T') = 1 + \max\{h(T'_L), h(T'_M), h(T'_R)\}$.
So,

$$\frac{3^{h(T')+1} - 1}{2} = \frac{3^{1+\max\{h(T'_L),h(T'_M),h(T'_R)\}+1} - 1}{2}.$$

$$= \frac{3^{1+\max\{h(T'_L),h(T'_M),h(T'_R)\}+1} - 3 + 2}{2}.$$

$$= \frac{3(3^{\max\{h(T'_L),h(T'_M),h(T'_R)\}+1} - 1) + 2}{2}.$$

$$= \frac{3(3^{\max\{h(T'_L),h(T'_M),h(T'_R)\}+1} - 1)}{2} + 1.$$

By The Inductive Hypothesis,

$$N(T') = N(T'_L) + N(T'_M) + N(T'_R) + 1 \leq$$

$$3\left(\frac{3^{\max\{h(T'_L),h(T'_M),h(T'_R)\}+1} - 1}{2}\right) + 1$$

$$= \frac{3^{h(T')+1} - 1}{2}.$$