START RECORDING

Algebraic / Transcendental Numbers

CMSC 250

Comparing Cardinalities

- Let A, B be sets.
- If there exists an injection (1-1 mapping) between A and B, but no surjection (onto mapping) from A to B, we will say that $|A| < |B|$

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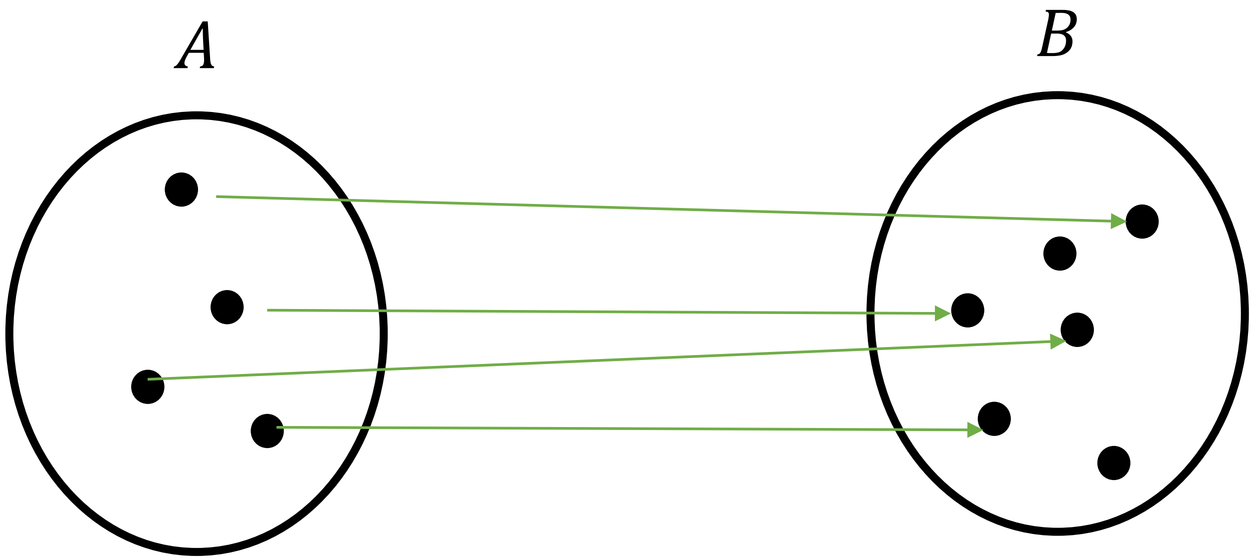
Here's an injection…



Comparing Cardinalities

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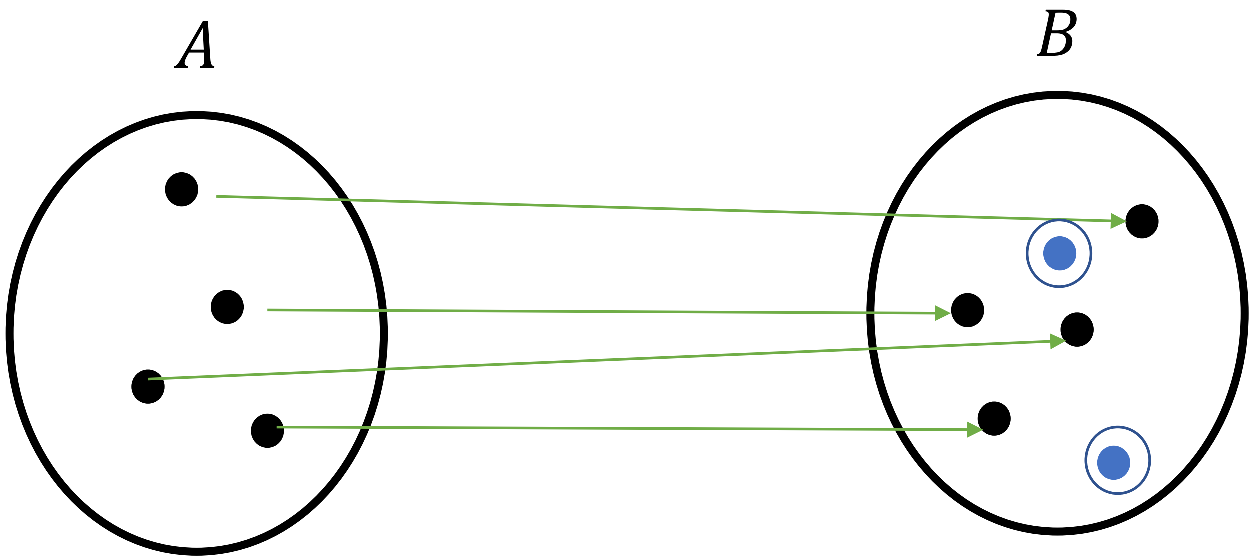
Here's an injection…



But there's no surjection ☹

Re-Define Rationals

- A rational is the root of an equation of the form

$$a \cdot x + b = 0$$

where $a, b \in \mathbb{Z}$.

- Also called **algebraic numbers of degree 1 (ALG1)**
  - Note: ALG1 is countable.

ALG2

- A number is in ALG2 if it is a root of an equation of the form

$$a \cdot x^2 + b \cdot x + c = 0$$

where $a, b, c \in \mathbb{Z}$

Examples of Numbers in ALG2

- 3 is a root of $x^2 - 9 = 0$
- $\sqrt{2}$ is a root of $x^2 - 2 = 0$ (so irrationals can be in ALG2!)
- $-\sqrt{2}$ is a root of $x^2 - 2 = 0$
- i is a root of $x^2 + 1 = 0$ (so complex numbers can be in ALG2!)
- $3i + 1$ is a root of $x^2 - 2x + 10 = 0$ (convince yourselves)

ALG2

- Recall: a number is in ALG2 if it is a root of an equation of the form

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- Is ALG2 countable?

Yes

No

Unknown to science

ALG2

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ALG2 Caveat (Countability Proof Follows)

1. Yes, ALG2 does contain some irrationals, e.g $\sqrt{5}$
2. ALG2 does not contain all of the reals. There are notes on the class slides website to show that $2^{1/3}$ is not in ALG2. The proof requires linear algebra. It is not hard; however, it is not required for this course.
3. ALG3 (you can guess) does not contain all of the reals. There are notes on the class website to show that $2^{1/4}$ is not in ALG3. This proof also requires linear algebra. It is also not hard; however, it is not required for this course.
4. Key: there aren't "that many" irrationals in ALG2.

ALG2 is Countable

1. We identify $a \cdot x^2 + b \cdot x + c = 0$ with triple (a, b, c)
2. Recall: $(a, b, c) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.
3. Recall: $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is countable.
4. So we can list out all quadratic equations as q_1, q_2, q_3 ...
   - Let r_{11}, r_{12} be roots of q_1,
   - Let r_{21}, r_{22} be roots of q_2,
   - …
   - Let r_{i1}, r_{i2} be roots of q_i,
5. List of roots: $r_{11}, r_{12}, r_{21}, r_{22}, r_{31}, r_{32}, \ldots$

ALG2 is Countable

- List of roots: $r_{11}, r_{12}, r_{21}, r_{22}, r_{31}, r_{32}, \ldots$
- This shows that ALG2 is countable
- Caveat: some roots might appear more than once in the list.
- 2 solutions:
    1. Just remove them (like in the proof that $\mathbb{Q}$ is countable)
    2. Theorem: subset of countable set is countable. (prove it yourselves)

ALG3

- A number is in ALG3 if it is a root of an equation of the form

$$a \cdot x^3 + b \cdot x^2 + c \cdot x + d = 0$$

where $a, b, c, d \in \mathbb{Z}$

ALG3

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2. Recall: $(a, b, c, d) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.
3. Recall: $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is countable.
4. Can list out all cubic equations as q_1, q_2, q_3 ...
   - Let r_{11}, r_{12}, r_{13} be roots of q_1,
   - Let r_{21}, r_{22}, r_{23} be roots of q_2,
   - ...
   - Let r_{i1}, r_{i2}, r_{i3} be roots of q_i,
5. List of roots: $r_{11}, r_{12}, r_{13}, r_{21}, r_{22}, r_{23}, r_{31}, r_{32}, r_{33}, \ldots$

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Same argument:
ALG3 is countable

ALGi Countable ($i \in \mathbb{N}$)

- We prove this:

ALGi Countable ($i \in \mathbb{N}$)

- We prove this:
- NOPE!
  - We are busy people (class moto)

The Algebraic Numbers

- Definition: A number is algebraic if it's a root of a polynomial with integer co-efficients.
- Denote the set ALG. Note that:

$$ALG = \bigcup_{i=1}^{+\infty} ALG_i$$

- Since union of countable sets is countable and each ALG_i is countable, ALG is countable ☺

Definition

- A number is **transcendental** if it is does not satisfy any algebraic equation over the integers.
- Denote the set of transcendental numbers with TN
- $TN = \mathbb{C} - ALG$ (remember: $\mathbb{C}$ is the set of complex numbers).

Numbers in TN

- Can you name numbers in TN?

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  - Any more **actually** different?
  - $0.1\underbrace{0}_{1!\text{ zeroes}}1\underbrace{00}_{2!\text{ zeroes}}1\underbrace{000000}_{3!\text{ zeroes}}1\underbrace{000000000000000000000000}_{4!\text{ zeroes}}1 \ldots.$

  - "We" (Bill) can prove that such numbers are not algebraic.
    - The proof of this will not be in the final, unless..

Any Other Numbers in TN?

- Are there any other numbers in TN?

Yes

No

Unknown to science

Any Other Numbers in TN?

- Are there any other numbers in TN?

Yes

No

Unknown to science

- Can we name any?

Any Other Numbers in TN?

- Are there any other numbers in TN?

Yes | No | Unknown to science

- Can we name any?
  - NO ☹

Any Other Numbers in TN?

- Are there any other numbers in TN?

Yes

No

Unknown to science

- Can we name any?
  - NO ☹
  - But hold on! We will talk about this matter soon. ☺

Size of TN

- Before we look inside TN any further, **is it countable?**

Yes

No

Unknown to science

Size of TN

- Before we look inside TN any further, **is it countable?**

Yes	No	Unknown to science

- Recall:
  1. $TN = \mathbb{C} - ALG$ (**T**ransce**N**dental numbers are all non-**ALG**ebraic complex numbers)
  2. $\mathbb{C}$ is uncountable (countability lecture)
  3. ALG countable
  - From 1, 2 and 3 we can deduce that TN is uncountable

Punchline

- **Most** numbers are transcendental!
- But most numbers **we (humans) use** are **not**!
- Recall the proof that there exist (uncountably many) transcendental numbers:
  1. ALG countable
  2. $\mathbb{C}$ is uncountable
  3. So $TN = \mathbb{C} - ALG$ is uncountable, hence $TN \neq \emptyset$
- This proof is **non-constructive**, since it does not produce a single transcendental number!

Punchline

- **Most** numbers are transcendental!
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- Recall the second proof, that there exist (uncountably many) transcendental numbers:
    1. ALG countable
    2. $\mathbb{C}$ is uncountable
    3. So $TN = \mathbb{C} - ALG$ is uncountable, hence $TN \neq \emptyset$
- This proof is **non-constructive**, since it does not produce a single transcendental number!
    - Hence, besides the ones we provided you with ($\pi, e, 0.1010010000001000000000000000000000000001 \ldots .$) we can't give you more!

New topic: Cardinality

Cardinality

- Recall: A and B **of the same size** if there's a **bijection from $\boldsymbol{A}$ to $\boldsymbol{B}$.**
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^{>0}, \mathbb{Q}^{<0}, \mathbb{Q}, \mathbb{N}^{even}, \mathbb{N}^{odd}, \mathbb{Z}^{even}, \mathbb{Z}^{odd}, \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are all of the same size

Cardinality

- Recall: A and B **of the same size** if there's a **bijection from $\boldsymbol{A}$ to $\boldsymbol{B}$.**
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^{>0}, \mathbb{Q}^{<0}, \mathbb{Q}, \mathbb{N}^{even}, \mathbb{N}^{odd}, \mathbb{Z}^{even}, \mathbb{Z}^{odd}, \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are all of the same size
  - This cardinality is denoted $\aleph_0$ (aleph-naught)

Cardinality

- What about $(0,1), [0,1], \mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathbb{R} \times \mathbb{R} \times \mathbb{R}$? **Are all of these the same size?**

Yes

No

Unknown to science

Cardinality

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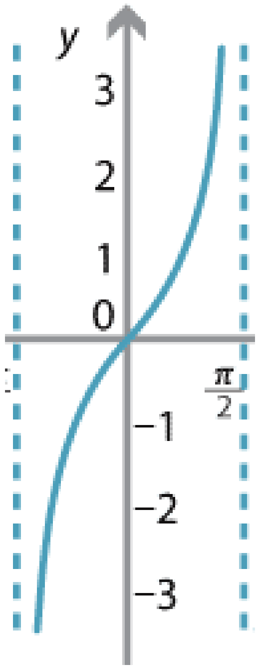
- Proof follows

$(-\frac{\pi}{2}, \frac{\pi}{2})$ Same Size as $\mathbb{R}$

- **Tangent function:** domain $(-\frac{\pi}{2}, \frac{\pi}{2})$, co-domain $\mathbb{R}$.
  - Both onto and 1-1, so bijection.



Bijection from $(0,1)$ to $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

- This linear function is a bijection from $(0,1)$ to $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$:

$$f(x) = \pi \cdot x - \frac{\pi}{2}$$

- So we have a bijection from $(0,1)$ to $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$…
  - … and a bijection from $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ to $\mathbb{R}$…
    - Which means that $(0,1)$, $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\mathbb{R}$ are all the same size!

$[0,1], (0,1], [0,1), (0,1)$

- All the same size.
- We are busy people and will not prove this.

$(0,1), (0,1) \times (0,1)$

- We define a bijection $f: (0,1) \mapsto (0,1) \times (0,1)$ as follows:

$$f(0.x_1x_2x_3x_4x_5x_6 \ldots) =$$
$$(0.x_1x_3x_5 \ldots, 0.x_2x_4x_6 \ldots,)$$

- Surprising, since $(0,1)$ is 1D and $(0,1) \times (0,1)$ is 2D.
- Bijections do **not necessarily** preserve dimension!

$(0,1), (0,1) \times (0,1) \times (0,1)$

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$$(0,1), (0,1) \times (0,1) \times (0,1)$$

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$$\Big(\underbrace{0.x_1x_4x_7}_{x_i \text{ with } i \equiv 1\ (mod\ 3)}\ldots, \underbrace{0.x_2x_5x_8}_{x_i \text{ with } i \equiv 2\ (mod\ 3)}\ldots, \underbrace{0.x_3x_6x_9}_{x_i \text{ with } i \equiv 2\ (mod\ 3)}\ldots\Big)$$

Explaining the Result of our Earlier Voting

- Recall that we now know $(0,1)$ same size as $\mathbb{R}$
- We have also established that $(0,1)$, $[0,1]$, $\mathbb{R}$, $\mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ **all the same size,** which explains the vote of "Yes".

Yes

No

Unknown to science

$\mathcal{P}(\mathbb{N}), \mathbb{R}$ Same Size?

- $\mathbb{R}$ uncountable
- $\mathcal{P}(\mathbb{N})$ uncountable
- **Are they the same size?**

Yes

No

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Digression: Real Numbers in Base 10

- Normally, reals are in base 10. Example:

$$3.14159\ldots = 3 \times 10^0 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5}$$

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- Why do we use base 10?

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- Why do we use base 10?

Digression: Real Numbers in Base 2

- Could just as easily express all reals in base 2.

$$11.0110\ldots = 1 \times 2^1 + 1 \times 2^0 + \frac{0}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{0}{10^4}$$

- So, all numbers in $[0, 1]$ are expressible as an **infinite sequence** of 0s and 1s in base 2.

Endpoints of $[0, 1]$

- Note that:

$$0.1111111_{(2)} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

Endpoints of $[0, 1]$

- Note that:

$$0.1111111_{(2)} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots \mapsto 1$$

Endpoints of $[0, 1]$

- Note that:

$$0.1111111_{(2)} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots \mapsto 1 \text{ (by convention, } = 1)$$

- Upshot: we view elements of $[0, 1]$ as infinite sequences of 0s and 1s

$\mathcal{P}(\mathbb{N}), \mathbb{R}$ Same Size!

- View $\mathcal{P}(\mathbb{N})$ as an infinite sequence of 0s and 1s
  - Let's see how this would work for $\mathbf{P}$, the set of primes:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	…
0	0	1	1	0	1	0	1	0	0	0	1	0	1	…

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0	0	1	1	0	1	0	1	0	0	0	1	0	1	…

We map this to

0. 0 0 1 1 0 1 0 1 0 0 0 1 0 1 …

which is a real number in (0, 1) expressed **in base 2!**

Bijection from $\mathcal{P}(\mathbb{N})$ to $[0,1]$

- $a_1 a_2 a_3 \ldots \in \{0,1\}^\omega$ (infinite sequences of 0s and 1s), hence an element of $\mathcal{P}(\mathbb{N})$

  maps to

  $0.a_1 a_2 a_3 \ldots$

  which is a real number in base 2.

Shorter Version

1. $[0,1]$ can be viewed as the set of all infinite sequences of 0s and 1s. $(\{0,1\}^{\omega})$
2. $\mathcal{P}(\mathbb{N})$ can also be viewed as the same set.
3. Hence, they are the same size.

Shorter Version

1. $[0, 1]$ can be viewed as the set of all infinite sequences of 0s and 1s. ($\{0, 1\}^{\omega}$)
2. $\mathcal{P}(\mathbb{N})$ can also be viewed as the same set.
3. Hence, they are the same size.

- Recall: $\{0, 1\}^{\omega}$ **uncountable.**

Between $\mathbb{N}$ and $\mathbb{R}$

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^{>0}, \mathbb{Q}^{<0}, \mathbb{Q}, \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are all of the same size
- $(0,1), [0,1], \mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ also of the same size
- $|\mathbb{N}| < |\mathbb{R}|$ (by diagonalization)

Between $\mathbb{N}$ and $\mathbb{R}$

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^{>0}, \mathbb{Q}^{<0}, \mathbb{Q}, , \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are all of the same size
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- $|\mathbb{N}| < |\mathbb{R}|$ (by diagonalization)
- Does there exist a set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$?

Yes

No

Unknown to science

Between $\mathbb{N}$ and $\mathbb{R}$

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^{>0}, \mathbb{Q}^{<0}, \mathbb{Q}, , \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are all of the same size
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Continuum Hypothesis

It's Actually Worse than Unknown!

- Let CH be the statement: "There is no set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$"
- ZFC is a set of 9 axioms from which you can derive all mathematics
  - Example: If A and B are sets, so is $A \cup B$, and so are $\mathcal{P}(A), \mathcal{P}(B)$,

1. $ZFC \cup CH$ does **not** lead to a contradiction.
2. $ZFC \cup (\sim CH)$ **also** does **not** lead to a contradiction!
3. Hence, CH will **never** be proven or disproven.

"Resolving" CH

- There are those who think CH can be resolved **by adding new axioms to Set Theory.**
- Bill says they're **stupid**, because the axioms are not **obviously true.**

Alephs

- Reminder: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^{>0}, \mathbb{Q}^{<0}, \mathbb{Q}, , \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are all of the same cardinality, denoted $\aleph_0$ (aleph-naught)
- $(0,1), [0,1], \mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathbb{R} \times \mathbb{R} \times \mathbb{R}, P(\mathbb{N})$ also of the same size.
- How do we denote the cardinality of those sets?

$\aleph_1$	$2^{\aleph_0}$	$\aleph_0 + 1$	Something else

Alephs

- Reminder: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^{>0}, \mathbb{Q}^{<0}, \mathbb{Q}, , \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are all of the same cardinality, denoted $\aleph_0$ (aleph-naught)
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Why is $|\mathbb{R}|$ not denoted $\aleph_1$?

- If there's no set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$ then $|\mathbb{R}| = \aleph_1$.
- If there is one such set, then $|\mathbb{R}| = \aleph_2$.
- If there are two such sets, then…

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  - We won't continue adding indices to $\aleph$, we are busy people.

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- If there are two such sets, then…
    - We won't continue adding indices to $\aleph$, we are busy people.
- We do not (and cannot) know which among those two holds, so can't use any $\aleph_i$ for $|\mathbb{R}|$.

Why is $|\mathbb{R}|$ denoted $2^{\aleph_0}$?

- $|\mathbb{R}|=|\mathcal{P}(\mathbb{N})|$
- Recall: For any $n \in \mathbb{N}$, $|\mathcal{P}(\{1,2,3,\dots,n\})| = 2^n$
- We extend this notation to $|\mathcal{P}(A)| = 2^{|A|}$.

Hence $|\mathcal{P}(\mathbb{N})| = 2^{|\mathbb{N}|} = 2^{\aleph_0}$

STOP

RECORDING