START
RECORDING
Algebraic / Transcendental Numbers

CMSC 250
Comparing Cardinalities

• Let $A, B$ be sets.

• If there exists an injection (1-1 mapping) between $A$ and $B$, but no surjection (onto mapping) from $A$ to $B$, we will say that $|A| < |B|$
Comparing Cardinalities

• Let \( A, B \) be sets.
• If there exists an injection (1-1 mapping) between \( A \) and \( B \), but no surjection (onto mapping) from \( A \) to \( B \), we will say that \( |A| < |B| \)

Here’s an injection...
Comparing Cardinalities

Let $A, B$ be sets.

If there exists an injection (1-1 mapping) between $A$ and $B$, but no surjection (onto mapping) from $A$ to $B$, we will say that $|A| < |B|$.
Re-Define Rationals

• A rational is the root of an equation of the form

\[ a \cdot x + b = 0 \]

where \( a, b \in \mathbb{Z} \).

• Also called *algebraic numbers of degree 1 (ALG1)*
  • Note: ALG1 is *countable*.
• A number is in ALG2 if it is a root of an equation of the form

\[ a \cdot x^2 + b \cdot x + c = 0 \]

where \( a, b, c \in \mathbb{Z} \)
Examples of Numbers in ALG2

- 3 is a root of $x^2 - 9 = 0$
- $\sqrt{2}$ is a root of $x^2 - 2 = 0$ (so irrationals can be in ALG2!)
- $-\sqrt{2}$ is a root of $x^2 - 2 = 0$
- $i$ is a root of $x^2 + 1 = 0$ (so complex numbers can be in ALG2!)
- $3i + 1$ is a root of $x^2 - 2x + 10 = 0$ (convince yourselves)
ALG2

• Recall: a number is in ALG2 if it is a root of an equation of the form

\[ a \cdot x^2 + b \cdot x + c = 0 \]

where \( a, b, c \in \mathbb{Z} \)

• Is ALG2 countable?

Yes  No  Unknown to science
ALG2

• Recall: a number is in ALG2 if it is a root of an equation of the form

\[ a \cdot x^2 + b \cdot x + c = 0 \]

where \( a, b, c \in \mathbb{Z} \)

• Is ALG2 countable?
  - Yes
  - No
  - Unknown to science
ALG2 Caveat (Countability Proof Follows)

1. Yes, ALG2 does contain some irrationals, e.g. $\sqrt{5}$

2. ALG2 does not contain all of the reals. There are notes on the class slides website to show that $2^{1/3}$ is not in ALG2. The proof requires linear algebra. It is not hard; however, it is not required for this course.

3. ALG3 (you can guess) does not contain all of the reals. There are notes on the class website to show that $2^{1/4}$ is not in ALG3. This proof also requires linear algebra. It is also not hard; however, it is not required for this course.

4. Key: there aren’t “that many” irrationals in ALG2.
1. We identify $a \cdot x^2 + b \cdot x + c = 0$ with triple $(a, b, c)$
2. Recall: $(a, b, c) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.
3. Recall: $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is countable.
4. So we can list out all quadratic equations as $q_1, q_2, q_3$ ...
   • Let $r_{11}, r_{12}$ be roots of $q_1$,
   • Let $r_{21}, r_{22}$ be roots of $q_2$,
   • ...
   • Let $r_{i1}, r_{i2}$ be roots of $q_i$,
5. List of roots: $r_{11}, r_{12}, r_{21}, r_{22}, r_{31}, r_{32}, ...$
ALG2 is Countable

• List of roots: \( r_{11}, r_{12}, r_{21}, r_{22}, r_{31}, r_{32}, \ldots \)

• This shows that \textbf{ALG2 is countable}

• Caveat: \textit{some roots might appear more than once in the list.}

• 2 solutions:
  1. Just \textbf{remove them} (like in the proof that \( \mathbb{Q} \) is countable)
  2. Theorem: \textit{subset of countable set is countable}. (prove it yourselves)
ALG3

• A number is in **ALG3** if it is a root of an equation of the form

\[ a \cdot x^3 + b \cdot x^2 + c \cdot x + d = 0 \]

where \( a, b, c, d \in \mathbb{Z} \)
• A number is in ALG3 if it is a root of an equation of the form

\[ a \cdot x^3 + b \cdot x^2 + c \cdot x + d = 0 \]

where \( a, b, c, d \in \mathbb{Z} \)

• Is ALG3 countable?
  - Yes
  - No
  - Unknown to science
ALG3

• A number is in ALG3 if it is a root of an equation of the form

\[ a \cdot x^3 + b \cdot x^2 + c \cdot x + d = 0 \]

where \( a, b, c, d \in \mathbb{Z} \)

• Is ALG3 countable? [Yes, No, Unknown to science]
ALG3 is Countable

1. We identify \( a \cdot x^3 + b \cdot x^2 + c \cdot x + d = 0 \) with triple \((a, b, c, d)\).
2. Recall: \((a, b, c, d) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\).
3. Recall: \(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\) is countable.
4. Can list out all cubic equations as \(q_1, q_2, q_3\) ...
   - Let \(r_{11}, r_{12}, r_{13}\) be roots of \(q_1\),
   - Let \(r_{21}, r_{22}, r_{23}\) be roots of \(q_2\),
   - ...
   - Let \(r_{i1}, r_{i2}, r_{i3}\) be roots of \(q_i\),
5. List of roots: \(r_{11}, r_{12}, r_{13}, r_{21}, r_{22}, r_{23}, r_{31}, r_{32}, r_{33}, \ldots\)
ALG3 is Countable

1. We identify $a \cdot x^3 + b \cdot x^2 + c \cdot x + d = 0$ with triple $(a, b, c, d)$

2. Recall: $(a, b, c, d) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. 

3. Recall: $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is countable. 

4. Can list out all cubic equations as $q_1, q_2, q_3 \ldots$
   - Let $r_{11}, r_{12}, r_{13}$ be roots of $q_1$,
   - Let $r_{21}, r_{22}, r_{23}$ be roots of $q_2$,
   - ...
   - Let $r_{i1}, r_{i2}, r_{i3}$ be roots of $q_i$,

5. List of roots: $r_{11}, r_{12}, r_{13}, r_{21}, r_{22}, r_{23}, r_{31}, r_{32}, r_{33}, \ldots$

Same argument: ALG3 is countable
ALG $i$ Countable ($i \in \mathbb{N}$)

• We prove this:
ALG\textsubscript{i} Countable (i ∈ \mathbb{N})

• We prove this:
  • NOPE!
    • We are busy people (class moto)
The Algebraic Numbers

• Definition: A number is algebraic if it’s a root of a polynomial with integer co-efficients.

• Denote the set $ALG$. Note that:

\[
ALG = \bigcup_{i=1}^{+\infty} ALG_i
\]

• Since union of countable sets is countable and each $ALG_i$ is countable, $ALG$ is countable ☺️
Definition

• A number is **transcendental** if it is does not satisfy any algebraic equation over the integers.

• Denote the set of transcendental numbers with $TN$

• $TN = \mathbb{C} - ALG$ (remember: $\mathbb{C}$ is the set of complex numbers).
Numbers in $TN$

- Can you name numbers in $TN$?
Numbers in $TN$

• Can you name numbers in $TN$?
  • $\pi$ (this is a hard theorem, says Bill)
Numbers in $TN$

• Can you name numbers in $TN$?
  • $\pi$ (this is a hard theorem, says Bill)
  • $e$ (easier but still hard)
Numbers in $TN$

• Can you name numbers in $TN$?
  • $\pi$ (this is a hard theorem, says Bill)
  • $e$ (easier but still hard)
  • Any more?
Numbers in $TN$

• Can you name numbers in $TN$?
  • $\pi$ (this is a hard theorem, says Bill)
  • $e$ (easier but still hard)
  • Any more?
  • “$2\pi$”, says Jason (Don’t be a wiseguy, says Bill)
Numbers in $TN$

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  • Any more actually different?
Numbers in $TN$

• Can you name numbers in $TN$?
  • $\pi$ (this is a hard theorem, says Bill)
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  • Any more actually different?
  • $0.10100100000100000000000000000001\ldots$
Numbers in $TN$

- Can you name numbers in $TN$?
  - $\pi$ (this is a hard theorem, says Bill)
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  - Any more?
  - “2$\pi$”, says Jason (“Don’t be a wiseguy”, says Bill)
  - Any more actually different?
  - 0.10100100000000000000000000000000000001 ….
Numbers in $TN$

- Can you name numbers in $TN$?
  - $\pi$ (this is a hard theorem, says Bill)
  - $e$ (easier but still hard)
  - Any more?
  - “$2\pi$”, says Jason (“Don’t be a wiseguy”, says Bill)
  - Any more actually different?
  - $0.10100100000010000000000000000000000001 \ldots$
    - 1! zeroes
    - 2! zeroes
    - 3! zeroes
    - 4! zeroes

- “We” (Bill) can prove that such numbers are not algebraic.
  - The proof of this will not be in the final, unless..
Any Other Numbers in $TN$?

• Are there any other numbers in $TN$?

Yes  No  Unknown to science
Any Other Numbers in $TN$?

- Are there any other numbers in $TN$?

- Can we name any?

- Yes
- No
- Unknown to science
Any Other Numbers in $TN$?

• Are there any other numbers in $TN$?
  
  ![Yes](blue) ![No](green) ![Unknown to science](yellow)

• Can we name any?
  • NO 😞
Any Other Numbers in $TN$?

• Are there any other numbers in $TN$?
  - Yes
  - No
  - Unknown to science

• Can we name any?
  • NO 😞
  • But hold on! We will talk about this matter soon. 😊
Size of $TN$

• Before we look inside $TN$ any further, is it countable?

Yes  No  Unknown to science
Size of $TN$

• Before we look inside $TN$ any further, is it countable?

  Yes  No  Unknown to science

• Recall:
  1. $TN = \mathbb{C} - ALG$ (Transcendental numbers are all non-ALGebraic complex numbers)
  2. $\mathbb{C}$ is uncountable (countability lecture)
  3. $ALG$ countable

  • From 1, 2 and 3 we can deduce that $TN$ is uncountable
• **Most** numbers are transcendental!

• But most numbers we (humans) use are **not**!

• Recall the proof that there exist (uncountably many) transcendental numbers:
  1. $ALG$ countable
  2. $\mathbb{C}$ is uncountable
  3. So $TN = \mathbb{C} - ALG$ is uncountable, hence $TN \neq \emptyset$

• This proof is **non-constructive**, since it does not produce a single transcendental number!
Most numbers are transcendental!

But most numbers we (humans) use are not!

Recall the second proof, that there exist (uncountably many) transcendental numbers:

1. \( \mathbb{ALG} \) countable
2. \( \mathbb{C} \) is uncountable
3. So \( TN = \mathbb{C} - \mathbb{ALG} \) is uncountable, hence \( TN \neq \emptyset \)

This proof is non-constructive, since it does not produce a single transcendental number!

Hence, besides the ones we provided you with

\( \pi, e, 0.10100100000010000000000000000000000000000001 \ldots \) we can’t give you more!
New topic: Cardinality
Cardinality

• Recall: $A$ and $B$ of the same size if there’s a bijection from $A$ to $B$.

• $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^0, \mathbb{Q}^{<0}, \mathbb{Q}, \mathbb{N}^{\text{even}}, \mathbb{N}^{\text{odd}}, \mathbb{Z}^{\text{even}}, \mathbb{Z}^{\text{odd}}, \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are all of the same size
Cardinality

• Recall: $A$ and $B$ of the same size if there’s a bijection from $A$ to $B$.

• $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^{>0}, \mathbb{Q}^{<0}, \mathbb{Q}, \mathbb{N}^{even}, \mathbb{N}^{odd}, \mathbb{Z}^{even}, \mathbb{Z}^{odd}, \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are all of the same size
  • This cardinality is denoted $\aleph_0$ (aleph-naught)
Cardinality

• What about \((0,1), [0, 1], \mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathbb{R} \times \mathbb{R} \times \mathbb{R}\)? Are all of these the same size?

Yes  No  Unknown to science
Cardinality

• What about $(0, 1)$, $[0, 1]$, $\mathbb{R}$, $\mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$? Are all of these the same size?

- Yes
- No
- Unknown to science

• Proof follows
\((-\frac{\pi}{2}, \frac{\pi}{2})\) Same Size as \(\mathbb{R}\)

- **Tangent function**: domain \((-\frac{\pi}{2}, \frac{\pi}{2})\), co-domain \(\mathbb{R}\).
  - Both onto and 1-1, so bijection.
Bijection from $(0, 1)$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

• This linear function is a bijection from $(0,1)$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$f(x) = \pi \cdot x - \frac{\pi}{2}$$

• So we have a bijection from $(0, 1)$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$...

• ... and a bijection from $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to $\mathbb{R}$...

• Which means that $(0, 1)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\mathbb{R}$ are all the same size!
[0, 1], (0, 1], [0, 1), (0, 1)

- All the same size.
- We are busy people and will not prove this.
We define a bijection $f: (0, 1) \mapsto (0, 1) \times (0, 1)$ as follows:

$$f(0.x_1x_2x_3x_4x_5x_6\ldots) = (0.x_1x_3x_5\ldots, 0.x_2x_4x_6\ldots)$$

Surprising, since $(0, 1)$ is 1D and $(0, 1) \times (0, 1)$ is 2D.

Bijections do not necessarily preserve dimension!
(0, 1), (0, 1) × (0, 1) × (0, 1)

• We define a bijection \( f: (0, 1) \mapsto (0, 1) \times (0, 1) \times (0, 1) \) as follows:

\[
f(0.x_1x_2x_3 \ldots) = (0.x_1x_4x_7 \ldots, 0.x_2x_5x_8 \ldots, 0.x_3x_6x_9 \ldots)
\]
We define a bijection $f: (0, 1) \mapsto (0, 1) \times (0, 1) \times (0, 1)$ as follows:

$$f(0.x_1x_2x_3 \ldots) = (0.x_1x_4x_7 \ldots, 0.x_2x_5x_8 \ldots, 0.x_3x_6x_9 \ldots)$$

$x_i$ with $i \equiv 1 \pmod{3}$  $x_i$ with $i \equiv 2 \pmod{3}$  $x_i$ with $i \equiv 2 \pmod{3}$
Explaining the Result of our Earlier Voting

• Recall that we now know \((0,1)\) same size as \(\mathbb{R}\)
• We have also established that \((0,1), [0, 1], \mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathbb{R} \times \mathbb{R} \times \mathbb{R}\) all the same size, which explains the vote of “Yes”.

Yes  No  Unknown to science
\[ \mathcal{P}(\mathbb{N}), \mathbb{R} \text{ Same Size?} \]

- \( \mathbb{R} \) uncountable
- \( \mathcal{P}(\mathbb{N}) \) uncountable
- Are they the same size?

Yes  |  No  |  Unknown to science
\[ \mathcal{P}(\mathbb{N}), \mathbb{R} \text{ Same Size!} \]

- \( \mathbb{R} \) uncountable
- \( \mathcal{P}(\mathbb{N}) \) uncountable
- Are they the same size?

- Yes
- No
- Unknown to science
Digression: Real Numbers in Base 10

- Normally, reals are in base 10. Example:

\[ 3.14159 \ldots = 3 \times 10^0 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} \]
Digression: Real Numbers in Base 10

• Normally, reals are in base 10. Example:

\[ 3.14159 \ldots = 3 \times 10^0 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} \]

• Why do we use base 10?
Digression: Real Numbers in Base 10

• Normally, reals are in base 10. Example:

\[ 3.14159 \ldots = 3 \times 10^0 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} \]

• Why do we use base 10?
Digression: Real Numbers in Base 2

- Could just as easily express all reals in base 2.

\[ 11.0110 \ldots = 1 \times 2^1 + 1 \times 2^0 + \frac{0}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{0}{10^4} \]

- So, all numbers in [0, 1] are expressible as an infinite sequence of 0s and 1s in base 2.
Endpoints of $[0, 1]$

• Note that:

\[ 0.1111111(2) = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots \]
Endpoints of $[0, 1]$

• Note that:

$$0.1111111_{(2)} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots \Rightarrow 1$$
Endpoints of $[0, 1]$ 

- Note that:

$$0.11111111_{(2)} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots \mapsto 1 \text{ (by convention, } = 1)$$

- **Upshot:** we view elements of $[0, 1]$ as infinite sequences of 0s and 1s
\[ \mathcal{P}(\mathbb{N}), \mathbb{R} \text{ Same Size!} \]

- View \( \mathcal{P}(\mathbb{N}) \) as an infinite sequence of 0s and 1s

- Let’s see how this would work for \( \mathbb{P} \), the set of primes:

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | ...
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|...
| 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |...
\( \mathcal{P}(\mathbb{N}), \mathbb{R} \) Same Size!

- View \( \mathcal{P}(\mathbb{N}) \) as an infinite sequence of 0s and 1s

- Let’s see how this would work for \( \mathbb{P} \), the set of primes:

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
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<th>3</th>
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<th>12</th>
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</thead>
<tbody>
<tr>
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<td>1</td>
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<td>1</td>
</tr>
</tbody>
</table>

We map this to

0. 0 0 1 1 0 1 0 1 0 0 0 0 1 0 1 ... 

which is a real number in (0, 1) expressed in base 2!
Bijection from $\mathcal{P}(\mathbb{N})$ to $[0, 1]$

- $a_1a_2a_3 \ldots \in \{0, 1\}^\omega$ (infinite sequences of 0s and 1s), hence an element of $\mathcal{P}(\mathbb{N})$

maps to

$0.a_1a_2a_3 \ldots$

which is a real number in base 2.
Shorter Version

1. [0, 1] can be viewed as the set of all infinite sequences of 0s and 1s. \((\{0, 1\}^\omega)\)

2. \(\mathcal{P}(\mathbb{N})\) can also be viewed as the same set.

3. Hence, they are the same size.
1. $[0, 1]$ can be viewed as the set of all infinite sequences of 0s and 1s. ($\{0, 1\}^\omega$)

2. $\mathcal{P}(\mathbb{N})$ can also be viewed as the same set.

3. Hence, they are the same size.

• Recall: $\{0, 1\}^\omega$ uncountable.
Between \( \mathbb{N} \) and \( \mathbb{R} \)

- \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}^>0, \mathbb{Q}^<0, \mathbb{Q}, \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) are all of the same size
- \( (0,1), [0, 1], \mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) also of the same size
- \( |\mathbb{N}| < |\mathbb{R}| \) (by diagonalization)
Between $\mathbb{N}$ and $\mathbb{R}$

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^{>0}, \mathbb{Q}^{<0}, \mathbb{Q}, \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are all of the same size
- $(0,1), [0,1], \mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ also of the same size
- $|\mathbb{N}| < |\mathbb{R}|$ (by diagonalization)
- Does there exist a set $A$ such that $|\mathbb{N}| < |A| < |\mathbb{R}|$?

- Yes
- No
- Unknown to science
Between \( \mathbb{N} \) and \( \mathbb{R} \)

- \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}^+, \mathbb{Q}^-, \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) are all of the same size
- \( (0,1), [0,1], \mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) also of the same size
- \( |\mathbb{N}| < |\mathbb{R}| \) (by diagonalization)
- Does there exist a set \( A \) such that \( |\mathbb{N}| < |A| < |\mathbb{R}| \)?

Yes  No  Unknown to science

Continuum Hypothesis
It’s Actually Worse than Unknown!

• Let CH be the statement: “There is no set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$”
• ZFC is a set of 9 axioms from which you can derive all mathematics
  • Example: If $A$ and $B$ are sets, so is $A \cup B$, and so are $\mathcal{P}(A)$, $\mathcal{P}(B)$,

1. $ZFC \cup CH$ does not lead to a contradiction.
2. $ZFC \cup (\sim CH)$ also does not lead to a contradiction!
3. Hence, $CH$ will never be proven or disproven.
“Resolving” CH

• There are those who think CH can be resolved by adding new axioms to Set Theory.
• Bill says they’re stupid, because the axioms are not obviously true.
Alephs

• Reminder: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^{>0}, \mathbb{Q}^{<0}, \mathbb{Q},, \mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are all of the same cardinality, denoted $\aleph_0$ (aleph-naught)

• $(0,1), [0,1], \mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathbb{R} \times \mathbb{R} \times \mathbb{R}, P(\mathbb{N})$ also of the same size.

• How do we denote the cardinality of those sets?

- $\aleph_1$
- $2^{\aleph_0}$
- $\aleph_0 + 1$
- Something else
Alephs

- Reminder: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}^{>0}$, $\mathbb{Q}^{<0}$, $\mathbb{Q}$, $\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ are all of the same cardinality, denoted $\aleph_0$ (aleph-naught).
- $(0,1)$, $[0,1]$, $\mathbb{R}$, $\mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $P(\mathbb{N})$ also of the same size.
- How do we denote the cardinality of those sets?

$\aleph_1$  $2^{\aleph_0}$  $\aleph_0 + 1$  Something else
Why is $|\mathbb{R}|$ not denoted $\aleph_1$?

- If there’s no set $A$ such that $|\mathbb{N}| < |A| < |\mathbb{R}|$ then $|\mathbb{R}| = \aleph_1$.
- If there is one such set, then $|\mathbb{R}| = \aleph_2$.
- If there are two such sets, then...
Why is $|\mathbb{R}|$ not denoted $\aleph_1$?

- If there’s no set $A$ such that $|\mathbb{N}| < |A| < |\mathbb{R}|$ then $|\mathbb{R}| = \aleph_1$.
- If there is one such set, then $|\mathbb{R}| = \aleph_2$.
- If there are two such sets, then...
  - We won’t continue adding indices to $\aleph$, we are busy people.
Why is $|\mathbb{R}|$ not denoted $\aleph_1$?

- If there’s no set $A$ such that $|\mathbb{N}| < |A| < |\mathbb{R}|$ then $|\mathbb{R}| = \aleph_1$.
- If there is one such set, then $|\mathbb{R}| = \aleph_2$.
- If there are two such sets, then...
  - We won’t continue adding indices to $\aleph$, we are busy people.
- We do not (and cannot) know which among those two holds, so can’t use any $\aleph_i$ for $|\mathbb{R}|$. 
Why is $|\mathbb{R}|$ denoted $2^{\aleph_0}$?

- $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$
- Recall: For any $n \in \mathbb{N}$, $|\mathcal{P}\left(\{1, 2, 3, \ldots, n\}\right)| = 2^n$
- We extend this notation to $|\mathcal{P}(A)| = 2^{|A|}$.

Hence $|\mathcal{P}(\mathbb{N})| = 2^{|\mathbb{N}|} = 2^{\aleph_0}$