Arithmetic Mean–Geometric Mean–Inequalities
AM and GM

Def

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$$\frac{x_1 + \cdots + x_n}{n}.$$
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(x_1 \cdots x_n)^{1/n}.
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How do AM and GM compare when \(x_1, \ldots, x_n \in \mathbb{R}^+\)?
AM and GM: \( n = 2 \)

Assume \( x, y \in \mathbb{R}^+ \).

How do \( \frac{x+y}{2} \) and \( \sqrt{xy} \) compare?

Proof also reveals that they are equal IFF \( x = y \).
AM and GM: $n = 2$

Assume $x, y \in \mathbb{R}^+$. How do $\frac{x+y}{2}$ and $\sqrt{xy}$ compare? Discuss.

Proof also reveals that they are equal IFF $x = y$. Why $n = 2$? It will be the base case. And more!
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$$\frac{x + y}{2} \geq \sqrt{xy}$$
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Square both sides
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\frac{x^2 - 2xy + y^2}{4} \geq 0
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\[
\frac{(x - y)^2}{4} \geq 0
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Proof also reveals that they are equal IFF \( x = y \).

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And more!
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$$(x - y)^2 \geq 0$$

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The AM-GM Theorem

**Thm** For all $n \in \mathbb{N}$ and for all $x_1, \ldots, x_n \in \mathbb{R}^+$

\[
\frac{x_1 + \cdots + x_n}{n} \geq (x_1 \cdots x_n)^{1/n}
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The AM-GM Theorem

**Thm** For all \( n \in \mathbb{N} \) and for all \( x_1, \ldots, x_n \in \mathbb{R}^+ \)

\[
\frac{x_1 + \cdots + x_n}{n} \geq (x_1 \cdots x_n)^{1/n}
\]

Equality happens iff \( x_1 = \cdots = x_n \).
Recall To prove $(\forall n \geq 2)[P(n)]$ by induction you prove
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Recall To prove \((\forall n \geq 2)[P(n)]\) by induction you prove
\[P(2)\]
\[(\forall n \geq 2))[P(n) \rightarrow P(n + 1)].\]
Recall To prove $(\forall n \geq 2)[P(n)]$ by induction you prove $P(2)$ 
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From these two you can get to any $n \geq 2$. 
Recall To prove \((\forall n \geq 2)[P(n)]\) by induction you prove
\(P(2)\)
\((\forall n \geq 2))[P(n) \rightarrow P(n + 1)].\)
From these two you can get to any \(n \geq 2.\)
Any set of rules that allows you to get to any number would work.
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Recall To prove $(\forall n \geq 2)[P(n)]$ by induction you prove
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We will prove

\(P(2) \) (we already did this).

\((\forall n)[P(2^{n-1}) \rightarrow P(2^n)]\)
Recall To prove \((\forall n \geq 2)[P(n)]\) by induction you prove
\[P(2)\]
\((\forall n \geq 2))[P(n) \rightarrow P(n + 1)].\]
From these two you can get to any \(n \geq 2\).
Any set of rules that allows you to get to any number would work.

We will prove
\[P(2)\ (\text{we already did this}).\]
\[(\forall n)[P(2^{n-1}) \rightarrow P(2^n)]\]
\[(\forall n < m)[P(m) \rightarrow P(n)]\ (\text{YES, } n < m). \ NOT \ a \ typo!\)
Recall To prove \((\forall n \geq 2)[P(n)]\) by induction you prove

\[P(2)\]
\((\forall n \geq 2))[P(n) \rightarrow P(n + 1)]\).

From these two you can get to any \(n \geq 2\).

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We will prove

\[P(2)\] (we already did this).

\[(\forall n)[P(2^{n-1}) \rightarrow P(2^n)]\]

\[(\forall n < m)[P(m) \rightarrow P(n)]\] (YES, \(n < m\). NOT a typo!)

From these implications we easily obtain \((\forall n)[P(n)]\).
\( P(2^n-1) \implies P(2^n) \)

IH \[ \frac{\sum_{i=1}^{2^n-1} x_i}{2^{n-1}} \geq (\prod_{i=1}^{2^n-1} x_i)^{1/2^{n-1}} \]
$P(2^{n-1}) \implies P(2^n)$

IH  $\frac{\sum_{i=1}^{2^n-1} x_i}{2^{n-1}} \geq (\prod_{i=1}^{2^n-1} x_i)^{1/2^n-1}$

IS  $\frac{\sum_{i=1}^{2^n} x_i}{2^n} = \frac{\sum_{i=1}^{2^{n-1}} x_i}{2^n} + \frac{\sum_{i=2^{n-1}+1}^{2^n} x_i}{2^n} = \frac{1}{2} \left( \frac{\sum_{i=1}^{2^n-1} x_i}{2^{n-1}} + \frac{\sum_{i=2^n-1+1}^{2^n} x_i}{2^{n-1}} \right)$
\[ P(2^{n-1}) \implies P(2^n) \]

IH \[ \sum_{i=1}^{2^n-1} x_i \geq (\prod_{i=1}^{2^n-1} x_i)^{1/2^{n-1}} \]

IS
\[
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\]

\[
\geq \frac{1}{2} \left( \left( \prod_{i=1}^{2^{n-1}} x_i \right)^{1/2^{n-1}} + \left( \prod_{i=2^n-1+1} x_i \right)^{1/2^{n-1}} \right)
\]
\[ P(2^{n-1}) \iff P(2^n) \]

**IH** \[ \frac{\sum_{i=1}^{2^{n-1}} x_i}{2^{n-1}} \geq (\prod_{i=1}^{2^{n-1}} x_i)^{1/2^{n-1}} \]

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\geq \frac{1}{2} \left( (\prod_{i=1}^{2^{n-1}} x_i)^{1/2^{n-1}} + (\prod_{i=2^{n-1}+1}^{2^n} x_i)^{1/2^{n-1}} \right)
\]

Next Slide
\[ P(2^{n-1}) \implies P(2^n) \text{ (cont)} \]

\[
\geq \frac{1}{2} \left( \left( \prod_{i=1}^{2^{n-1}} x_i \right)^{1/2^{n-1}} + \left( \prod_{i=2^{n-1}+1}^{2^n} x_i \right)^{1/2^{n-1}} \right)
\]
\[ P(2^{n-1}) \quad \Longrightarrow \quad P(2^n) \quad (\text{cont}) \]

\[ \geq \frac{1}{2} \left( \left( \prod_{i=1}^{2^{n-1}} x_i \right)^{1/2^{n-1}} + \left( \prod_{i=2^{n-1}+1}^{2^n} x_i \right)^{1/2^{n-1}} \right) \]

**Note** This is AM of 2 numbers! We use AM-GM-2 on it!
\[ P(2^{n-1}) \implies P(2^n) \text{ (cont)} \]

\[ \geq \frac{1}{2} \left(( \prod_{i=1}^{2^{n-1}} x_i)^{1/2^{n-1}} + \left( \prod_{i=2^{n-1}+1}^{2^n} x_i \right)^{1/2^{n-1}} \right) \]

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\geq \frac{1}{2} \left( \left( \prod_{i=1}^{2^{n-1}} x_i \right)^{1/2^{n-1}} + \left( \prod_{i=2^{n-1}+1}^{2^n} x_i \right)^{1/2^{n-1}} \right)
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\]
\[ P(2^{n-1}) \implies P(2^n) \] (cont)

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\[ \geq \left( \prod_{i=1}^{2^n} x_i \right)^{1/2^{n-1}} = \left( \prod_{i=1}^{2^n} x_i \right)^{1/2^n}. \]
\[ n < m: \quad P(m) \iff P(n) \]

\[ \text{IH} \quad (\forall x_1, \ldots, x_m) \left[ \frac{\sum_{i=1}^m x_i}{m} \geq (\prod_{i=1}^m x_i)^{1/m} \right]. \]
\[ n < m: \quad P(m) \implies P(n) \]

**IH** \((\forall x_1, \ldots, x_m)[\frac{\sum_{i=1}^{m} x_i}{m} \geq (\prod_{i=1}^{m} x_i)^{1/m}]\).

**IS** We care about \(\frac{x_1 + \cdots + x_n}{n}\).
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We need \( x_{n+1}, \ldots, x_m \) so we can use IH.
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We need \(x_{n+1}, \ldots, x_m\) so we can use IH.

\[
x_{n+1} = \cdots = x_m = \frac{x_1 + \cdots + x_n}{n} = \alpha.
\]
$n < m$: $P(m) \implies P(n)$

**IH** $(\forall x_1, \ldots, x_m)[\frac{\sum_{i=1}^{m} x_i}{m} \geq (\prod_{i=1}^{m} x_i)^{1/m}]$.

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And now we begin the proof, starting with $\alpha$. 
\( n < m: \ P(m) \implies P(n) \)

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x_{n+1} = \cdots = x_m = \frac{x_1 + \cdots + x_n}{n} = \alpha.
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And now we begin the proof, starting with \(\alpha\).

\[
\alpha = \frac{x_1 + \cdots + x_n}{n} = \frac{m}{n}(x_1 + \cdots + x_n).
\]
$n < m$: $P(m) \implies P(n)$ (cont)

$$\alpha = \frac{x_1 + \cdots + x_n}{n} = \frac{m}{n} \left( x_1 + \cdots + x_n \right).$$
\( n < m: \ P(m) \iff P(n) \) (cont)

\[
\alpha = \frac{x_1 + \cdots + x_n}{n} = \frac{m}{n} \left( \frac{x_1 + \cdots + x_n}{m} \right).
\]

We want to write this as the mean of \( m \) elements.
n < m: \( P(m) \iff P(n) \) (cont)

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\]
\[ n < m: \quad P(m) \iff P(n) \quad (cont) \]

\[ \alpha = \frac{x_1 + \cdots + x_n}{n} = \frac{m(x_1 + \cdots + x_n)}{m} = \frac{m}{n}(x_1 + \cdots + x_n) \]

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\[ \frac{x_1 + \cdots + x_n}{n} = \frac{m}{n}(x_1 + \cdots + x_n) = \]

\[ \frac{x_1 + \cdots + x_n + \frac{m}{n}(x_1 + \cdots + x_n) - x_1 - \cdots - x_n}{m} = \]
\( n < m: \ P(m) \implies P(n) \) (cont)

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\[
\frac{x_1 + \cdots + x_n + \frac{m-n}{n}(x_1 + \cdots + x_n)}{m} = \frac{x_1 + \cdots + x_n + (m-n)\alpha}{m}
\]
\[ n < m: \ P(m) \iff P(n) \ (\text{cont}) \]

\[
\alpha = \frac{x_1 + \cdots + x_n + (m - n)\alpha}{m}
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$n < m$: $P(m) \implies P(n)$ (cont)

$$\alpha = \frac{x_1 + \cdots + x_n + (m - n)\alpha}{m}$$

We have the mean of $m$ numbers! We can use IH!
\( n < m: \ P(m) \iff P(n) \) (cont)

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\alpha = \frac{x_1 + \cdots + x_n + (m-n)\alpha}{m} \geq \left(\prod_{i=1}^{n} x_i\alpha^{m-n}\right)^{1/m}
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$n < m$: $P(m) \iff P(n)$ (cont)

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\]

\[
\alpha^m \geq ((\prod_{i=1}^{n} x_i)^{\alpha^{m-n}})
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\( n < m: \ P(m) \iff P(n) \) (cont)

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\[
\alpha^m \geq ((\prod_{i=1}^{n} x_i)\alpha^{m-n})
\]

Multiply both sides by \( \alpha^{n-m} \) to get
$n < m$: $P(m) \iff P(n)$ (cont)

\[ \alpha = \frac{x_1 + \cdots + x_n + (m - n)\alpha}{m} \]

We have the mean of $m$ numbers! We can use IH!

\[ \alpha = \frac{x_1 + \cdots + x_n + (m - n)\alpha}{m} \geq \left( \prod_{i=1}^{n} x_i \alpha^{m-n} \right)^{1/m} \]

\[ \alpha^m \geq \left( \prod_{i=1}^{n} x_i \alpha^{m-n} \right) \]

Multiply both sides by $\alpha^{n-m}$ to get

\[ \alpha^n \geq \prod_{i=1}^{n} x_i \]
$n < m$: $P(m) \implies P(n)$ (cont)

$$\alpha = \frac{x_1 + \cdots + x_n + (m - n)\alpha}{m}$$

We have the mean of $m$ numbers! We can use IH!

$$\alpha = \frac{x_1 + \cdots + x_n + (m - n)\alpha}{m} \geq \left(\prod_{i=1}^{n} x_i\alpha^{m-n}\right)^{1/m}$$

$$\alpha^m \geq \left(\prod_{i=1}^{n} x_i\alpha^{m-n}\right)$$

Multiply both sides by $\alpha^{n-m}$ to get

$$\alpha^n \geq \prod_{i=1}^{n} x_i$$

$$\alpha \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}$$
Why This Example?

This example is interesting since it uses a diff induction scheme.
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Base Case

You can reach any $n \in \mathbb{N}$, then $(\forall n)[P(n)]$. 
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you can reach any $n \in \mathbb{N}$, then $(\forall n)[P(n)]$. 