START
RECORDING
Constructive Induction

CMSC 250
Introductory Example

• We already know that

\[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} = \frac{n^2}{2} + \frac{n}{2} \]

• But how? Who told us this?

• This is not how math works; we usually do not know the answer ahead of time!
Making a Good Guess with Calculus

• Calculus tells us that (discrete) sums are approximations of (continuous) integrals.
• Then, we can observe that:

\[
\sum_{i=1}^{n} i \approx \int_{1}^{n} x \, dx = \frac{1}{2} n^2 + c, \quad c \in \mathbb{R}
\]

• So we know that the sum ought to be some quadratic function of \( n \).
Making a Good Guess with CS

• Another way to guess the quadratic form would be with plotting!
• Suppose \( f(n) = \sum_{i=1}^{n} i \). Then:
  
  • \( f(0) = \sum_{i=1}^{0} i = 0 \)
  • \( f(1) = \sum_{i=1}^{1} i = 1 \)
  • \( f(2) = \sum_{i=1}^{2} i = 1 + 2 = 3 \)
  • \( f(3) = \sum_{i=1}^{3} i = 1 + 2 + 3 = 6 \)
  • ...  
  • \( f(2) = \sum_{i=1}^{30} i = 1 + 2 + \cdots + 30 = 465 \)

• We can then fit a curve and see the quadratic curve by ourselves!
Making a Good Guess

• We saw that the sum is some quadratic polynomial. This is all we know!

• So $\sum_{i=1}^{n} i$ is some $\text{poly}(n)$ with degree 2, i.e

$$\sum_{i=1}^{n} i = An^2 + Bn + C, \quad A, B, C \in \mathbb{R}$$

• How to determine $A$, $B$, and $C$?
General Logic

• Solve as if you had an inductive proof (so IB, IH, IS)
• For every step, we will establish **conditions** on A, B, C **such that** the relevant step is correct.
  • Contrast this with **directly proving** that every step is correct.
Constant $C$

• IB: LHS is $\sum_{i=1}^{0} i = 0$. For RHS to be equal to LHS we need:

$$An^2 + Bn + C = 0 \Rightarrow C = 0$$

• So we already know that $C = 0$. 
Co-efficients $A, B$

- IH: Assume that the proposition holds for $n \geq 0$. Then:

$$\sum_{i=1}^{n} i = An^2 + Bn$$

- IS: We want to prove that

$$\left( \sum_{i=1}^{n} i = An^2 + Bn \right) \Rightarrow \left( \sum_{i=1}^{n+1} i = A(n+1)^2 + B(n+1) \right)$$
Co-efficients $A, B$

• IH: Assume that the proposition holds for $n \geq 0$. Then:

\[ \sum_{i=1}^{n} i = An^2 + Bn \]

• IS: We want to prove that

\[ \left( \sum_{i=1}^{n} i = An^2 + Bn \right) \Rightarrow \left( \sum_{i=1}^{n+1} i = A(n + 1)^2 + B(n + 1) \right) \]
Co-efficients $A, B$

\[\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n + 1) = An^2 + Bn + (n + 1)\]

- We have to equate this to $A(n + 1)^2 + B(n + 1)$, since this is what we’re trying to prove:

\[
An^2 + Bn + (n + 1) = A(n + 1)^2 + B(n + 1) \Rightarrow An^2 + Bn + (n + 1) = An^2 + 2An + A + Bn + B \Rightarrow n + 1 = 2An + (A + B)
\]
Co-efficients $A, B$

\[ n + 1 = 2An + (A + B) \]

- This is an equality between polynomials of $k$, so equating the co-efficients yields:

\[
\begin{align*}
1 &= 2A \\
A + B &= 1
\end{align*}
\]
Co-efficients $A, B$

\[ n + 1 = 2An + (A + B) \]

• This is an equality between polynomials in $n$, so equating the co-efficients yields:

\[ 1 = 2A \]
\[ A + B = 1 \]

• Note: The IS did not end up with **TRUE**, but with conditions on $A,B$ for it to be **TRUE**.
All Our Constraints

1. $C = 0$
2. $A + B = 1$
3. $2 \cdot A = 1$

• Algebra yields $A = B = \frac{1}{2}$, so:

$$\sum_{i=0}^{n} i = \frac{1}{2}n^2 + \frac{1}{2}n + 0 = \frac{n(n + 1)}{2}$$
What if Our Guess is Wrong (Over)?

1. Suppose we guess

\[ \sum_{i=1}^{n} i = A \cdot n^3 + B \cdot n^2 + C \cdot n + D \]

2. This still works, we will just find \( A = 0 \) (try it at home!)
What if Our Guess is Wrong (Under)?

1. Suppose we guess

\[ \sum_{i=1}^{n} i = A \cdot n + B \]

2. This does not work (infeasible equation), no \( A, B \in \mathbb{R} \) will satisfy the constraints (try it at home!)
Another Example (with Bounds!)

• Let $a$ be a sequence defined as follows:

\[
a_n = \begin{cases} 
2, & n = 0 \\
50, & n = 1 \\
10a_{n-1} + 3a_{n-2}, & n \geq 2
\end{cases}
\]

• Task: Find an upper bound for $a_n$. 
Another Example (with Bounds!)

• Let $a$ be a sequence defined as follows:

\[ a_n = \begin{cases} 
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• Task: Find an upper bound for $a_n$.
• What kind of inductive structure am I expecting?

[Buttons: Weak, Strong]
Another Example (with Bounds!)

• Let $a$ be a sequence defined as follows:

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2, & n = 0 \\
50, & n = 1 \\
10a_{n-1} + 3a_{n-2}, & n \geq 2 
\end{cases}$$

• Task: Find an upper bound for $a_n$.

• What kind of inductive structure am I expecting?

An inductive base with $> 1$ elements and a recursive rule with references to two prior terms hints towards strong induction...
Key Step

\[ a_n = \begin{cases} 
2, & n = 0 \\
50, & n = 1 \\
10a_{n-1} + 3a_{n-2}, & n \geq 2 
\end{cases} \]

Because of our experience with sequences like Fibonacci, Tribonacci that all have this form, we suspect:

\[ a_n \leq C \cdot D^n, \quad C, D \in \mathbb{R} \]
Constraints on $C$

- IB:
  - $a_0 \leq C \cdot D^0 \iff 2 \leq C$
  - $a_1 \leq C \cdot D^1 \iff 50 \leq C \cdot D$
Inductive Hypothesis

• IB:
  • $a_0 \leq C \cdot D^0 \iff 2 \leq C$
  • $a_1 \leq C \cdot D^1 \iff 50 \leq C \cdot D$

• IH: Let $n \geq 1$. Assume that $(\forall i \in \{0, 1, 2, \ldots, n\})[a_i \leq C \cdot D^i]$
Inductive Step

• IB:
  • $a_0 \leq C \cdot D^0 \iff 2 \leq C$
  • $a_1 \leq C \cdot D^1 \iff 50 \leq C \cdot D$

• IH: Let $n \geq 1$. Assume that $\forall i \in \{0, 1, 2, \ldots n\}, \ a_i \leq C \cdot D^i$.

• IS:
  $(\forall i \in \{0, 1, 2, \ldots n\})[a_i \leq C \cdot D^i] \Rightarrow (a_{n+1} \leq C \cdot D^{n+1})$
Inductive Step

- **IS:** 
  \[(\forall i \in \{0, 1, 2, \ldots, n\})[a_i \leq C \cdot D^i] \Rightarrow (a_{n+1} \leq C \cdot D^{n+1})\]

- From the definition of \(a\), we have \(a_{n+1} = 10a_n + 3a_{n-1}\). Therefore,

\[a_{n+1} = 10a_n + 3a_{n-1} \leq 10 \cdot C \cdot D^n + 3 \cdot C \cdot D^{n-1} \quad \text{(By IH)}\]

- Want \(10 \cdot C \cdot D^n + 3 \cdot C \cdot D^{n-1} \leq C \cdot D^{n+1}\)
Inductive Step

• Want

\[10 \cdot C \cdot D^n + 3 \cdot C \cdot D^{n-1} \leq C \cdot D^{n+1} \iff 10 \cdot D^n + 3 \cdot D^{n-1} \leq D^{n+1}\]

• Dividing both sides by \(D^{n-1}\) yields:

\[10D + 3 \leq D^2\]
All Constraints

1. $2 \leq C$
2. $50 \leq C \cdot D$
3. $10D + 3 \leq D^2$

• We deal with constraint 3 first.
  • Smallest $D \in \mathbb{R}^>0$ that satisfies it:
All Constraints

1. $2 \leq C$
2. $50 \leq C \cdot D$
3. $10D + 3 \leq D^2$

• We deal with constraint 3 first.
  • Smallest $D \in \mathbb{R}^>0$ that satisfies it: NO, WE ARE BUSY PEOPLE AND WE DON’T WANT TO SPEND TIME SOLVING $D^2 - 10D - 3 \geq 0$
  • Smallest $D \in \mathbb{N}$ that satisfies it: $D = \cdots ??? (FIND ONE REAL QUICK, PLZ)$
All Constraints

1. \(2 \leq C\)
2. \(50 \leq C \cdot D\)
3. \(10D + 3 \leq D^2\)

• We deal with constraint 3 first.
  • Smallest \(D \in \mathbb{R}^+\) that satisfies it: NO, WE ARE BUSY PEOPLE AND WE DON’T WANT TO SPEND TIME SOLVING \(D^2 - 10D - 3 \geq 0\)
  • Smallest \(D \in \mathbb{N}\) that satisfies it: \(D = \cdots ? ? ?\) (FIND ONE REAL QUICK, PLZ)

\[ D = 11 \text{ works!} \]
All Constraints

1. $2 \leq C$
2. $50 \leq C \cdot D$
3. $10D + 3 \leq D^2$

• Constraint (3) satisfied when $D \geq 11$ (just discussed)
• Since we want to find tight bounds for $a_n$, to minimize $C$, we select $D = 11$ and from constraint (2) we have: $50 \leq C \cdot 11 \iff C \geq 4.55 \Rightarrow C_{min} = 4.55$
All Constraints

1. $2 \leq C$
2. $50 \leq C \cdot D$
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• Constraint (3) satisfied when $D \geq 11$ (just discussed)
• Since we want to find tight bounds for $a_n$, to minimize $C$, we select $D = 11$ and from constraint (2) we have: $50 \leq C \cdot 11 \iff C \geq 4.55 \Rightarrow C_{\text{min}} = 4.55$
• Conclusion:

$$a_n \leq 4.55 \cdot 11^n$$
Work on This

• A slight modification on the previous sequence:

\[ a_n = \begin{cases} 
10, & n = 0 \\
50, & n = 1 \\
10a_{n-1} + 3a_{n-2}, & n \geq 2 
\end{cases} \]

• Assuming that we still suspect \( a_n \leq C \cdot D^n \), you solve for the new \( C, D \) right now!
Work on This

• A slight modification on the previous sequence:

\[ a_n = \begin{cases} 
10, & n = 0 \\
50, & n = 1 \\
10a_{n-1} + 3a_{n-2}, & n \geq 2 
\end{cases} \]

• Assuming that we still suspect \( a_n \leq C \cdot D^n \), solve for the new \( C, D \)!
• Your solution ought to be \( C = 10, D = 11 \). What do you observe?
Coin Problem

• In Celestia, there are only 7$c$ and 10$c$ coins.
• We want to find the least monetary amount payable exclusively with such coins!
• In quantifiers (all quantifications assumed over $\mathbb{N}$)

\[(\forall n \geq A)(\exists n_1, n_2)[n = 7n_1 + 10n_2]\]

• Goal: Find constraints on $A$ via constructive induction!
• IB: ???
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• IB: Defer for later!!!
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(\forall n \geq A)(\exists n_1, n_2)[n = 7n_1 + 10n_2]
\]

• Goal: Find constraints on \( A \) via constructive induction!
• IB: Defer for later!!!
• IH: Assume that for \( n \geq A \), \( (\exists n'_1, n'_2)[n = 7 \cdot n'_1 + 10n'_2] \)
Coin Problem (IS)

• From the IH we have $(\exists n'_1, n'_2)[n = 7 \cdot n'_1 + 10n'_2]$

• How can we add/remove coins to get another cent?
Coin Problem (IS)

• From the IH we have \((\exists n'_1, n'_2)[n = 7 \cdot n'_1 + 10n'_2]\)

• How can we add/remove coins to get another cent?

1. \(n'_2 \geq 2\) : Remove two 10c coins, add three 7c coins

\[
n + 1 = 7n'_1 + 10n'_2 + 1 = 7n'_1 + 10n'_2 + (21 - 20) = 7(n'_1 + 3) + 10(n'_2 - 2)
\]
Coin Problem (IS)

• From the IH we have \((\exists n_1', n_2')[n = 7 \cdot n_1' + 10n_2']\)
• How can we add/remove coins to get another cent?
  1. \(n_2' \geq 2\): Remove two 10c coins, add three 7c coins

\[
\begin{align*}
n + 1 &= 7n_1' + 10n_2' + 1 = 7n_1' + 10n_2' + (21 - 20) \\
&= 7(n_1' + 3) + 10(n_2' - 2)
\end{align*}
\]

  2. \(n_1' \geq 7\): Remove seven 7c coins, add five 10c coins

\[
\begin{align*}
n + 1 &= 7n_1' + 10n_2' + 1 = 7n_1' + 10n_2' + (50 - 49) \\
&= 7(n_1' - 7) + 10(n_2' + 5)
\end{align*}
\]
3. \((n'_1 \leq 6) \land (n'_2 \leq 1)\): Max value is \(6 \times 7 + 1 \times 10 = 52\), so \(n \leq 52\).
RECAP

• We’ve shown that if \( n \geq 53 \), then

\[
((\exists n_1', n_2')[n = 7 \cdot n_1' + 10n_2']) \Rightarrow ((\exists n_1'', n_2'')[n + 1 = 7 \cdot n_1'' + 10n_2''])
\]

• For which \( n \) do we know that \((\exists a, b \in \mathbb{N})[n = 7a + 10b]\)?

\[\forall n \geq 52 \quad \forall n \geq 53 \quad \text{Something Else}\]
• We’ve shown that if $n \geq 53$, then

\[
((\exists n'_1, n'_2)[n = 7 \cdot n'_1 + 10n'_2]) \Rightarrow ((\exists n''_1, n''_2)[n + 1 = 7 \cdot n''_1 + 10n''_2])
\]

• For which $n$ do we know that $((\exists a, b \in \mathbb{N})[n = 7a + 10b])$?

\[
\forall n \geq 52 \quad \forall n \geq 53 \quad \text{Something Else}
\]

Only the implication holds! We don’t have any hard truth (base) about whether it EVER holds.
Coin Problem (IS)

3. \((n'_1 \leq 6) \land (n'_2 \leq 1)\): Max value is \(6 \times 7 + 1 \times 10 = 52\), so \(n \leq 52\).

- Condition: \(A \geq 53\).
- **Now** I need a base case.
- \((\exists n''_1, n''_2 \in \mathbb{N})[53 = 7 \cdot n''_1 + 10n''_2]\)
Coin Problem (IS)

3. \((n_1' \leq 6) \land (n_2' \leq 1)\): Max value is \(6 \times 7 + 1 \times 10 = 52\), so \(k \leq 52\).

- Condition: \(A \geq 53\).
- **Now** I need a base case.
- \((\exists n_1'', n_2'' \in \mathbb{N})[53 = 7 \cdot n_1'' + 10n_2'']\)

Prove it at home (use cases)
Coin Problem (IS)

3. \((n_1' \leq 6) \land (n_2' \leq 1)\): Max value is \(6 \times 7 + 1 \times 10 = 52\), so \(k \leq 52\).

- Condition: \(A \geq 53\).
- **Now** I need a base case.
- \((\exists \, n_1'', n_2'' \in \mathbb{N})[53 = 7 \cdot n_1'' + 10n_2'']\)
- \((\exists \, n_1'', n_2'' \in \mathbb{N})[54 = 7 \cdot n_1'' + 10n_2'']\)
Coin Problem (IS)

3. \((n'_1 \leq 6) \land (n'_2 \leq 1)\): Max value is \(6 \times 7 + 1 \times 10 = 52\), so \(k \leq 52\).

- Condition: \(A \geq 53\).
- **Now** I need a base case.

\(- (\exists? n''_1, n''_2 \in \mathbb{N})[53 = 7 \cdot n''_1 + 10n''_2]\)

\(- (\exists? n''_1, n''_2 \in \mathbb{N})[54 = 7 \cdot n''_1 + 10n''_2]\)

\(n''_1 = 2, \quad n''_2 = 4\)
RECAP

• We’ve shown that if \( n \geq 53 \), then

\[
((\exists n_1', n_2')[n = 7 \cdot n_1' + 10n_2']) \Rightarrow ((\exists n_1'', n_2'')[n + 1 = 7 \cdot n_1'' + 10n_2''])
\]

• We’ve also shown that \((\exists r_1, r_2 \in \mathbb{N})[54 = 7r_1 + 10r_2] \quad (r_1 = 2, r_2 = 4)\)
We've shown that if $n \geq 53$, then

$$((\exists n_1', n_2')[n = 7 \cdot n_1' + 10n_2']) \Rightarrow ((\exists n_1'', n_2'')[n + 1 = 7 \cdot n_1'' + 10n_2''])$$

We've also shown that $(\exists r_1, r_2 \in \mathbb{N})[54 = 7r_1 + 10r_2]$,

$(r_1 = 2, r_2 = 4)$

What do we know **NOW** about the theorem?

- True for $n \geq 52$
- True for $n \geq 53$
- True for $n \geq 54$
- Nothing
RECAP

• We’ve shown that if $n \geq 53$, then

$$((\exists n_1', n_2')[n = 7 \cdot n_1' + 10n_2']) \implies ((\exists n_1'', n_2'')[n + 1 = 7 \cdot n_1'' + 10n_2''])$$

• We’ve also shown that

$$((\exists r_1, r_2 \in \mathbb{N})[54 = 7r_1 + 10r_2])$$

$$\quad (r_1 = 2, r_2 = 4)$$

• What do we know **NOW** about the theorem?

- True for $n \geq 52$
- True for $n \geq 53$
- True for $n \geq 54$
- Nothing
What is $A$?

• Recall the theorem (all quantifiers over $\mathbb{N}$):

$$(\forall n \geq A)(\exists n_1, n_2)[n = 7n_1 + 10n_2]$$

• Our goal was to find $A$.

• $A = 54$ works, and is optimal, since $A = 53$ does not work.
Question

• Is the theorem true for any $n \leq 53$?

Yes
(which?)

No
(Why?)
Question

• Is the theorem true for any $n \leq 53$?

0, 7, 10, 14, 17, 20, 21, 24, 27, 28, 30, 31, 34, 35, 37, 38, 40, 41, 42, 44, 45, 47, 48, 49, 50, 51, 52

• Note that there are gaps between these integers!
General Scenarios

• Once we establish

\[(\forall n \geq n_0)[P(n) \Rightarrow P(n + 1)]\]

we have two cases:

1. \(P(n_0)\) is true. Then, we have to go back and find the first \(a \in \mathbb{N}\) for which \(P(n_0 - a)\) is false. This means that \(A = n_0 - a + 1\)
General Scenarios

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General Scenarios

• Once we establish

$$(\forall n \geq n_0)[P(n) \Rightarrow P(n + 1)]$$

we have two cases:

1. $P(n_0)$ is true. Then, we have to go back and find the first $a \in \mathbb{N}$ for which $P(n_0 - a)$ is false. This means that $A = n_0 - a + 1$
General Scenarios

• Once we establish

\[(\forall n \geq n_0)[P(n) \Rightarrow P(n + 1)]\]

we have a second case:

2. \(P(n_0)\) is false. Then, we have to go forward and find the first \(a \in \mathbb{N}\) for which \(P(n_0 + a)\) is true. This means that \(A = n_0 + a\)
2. $P(n_0)$ is false. Then, we have to go forward and find the first $a \in \mathbb{N}$ for which $P(n_0 + a)$ is true.

a) We find $a$ such that $P(n_0 + a)$ is true.
2. $P(n_0)$ is false. Then, we have to go forward and find the first $a \in \mathbb{N}$ for which $P(n_0 + a)$ is true.

a) We find $a$ such that $P(n_0 + a)$ is true.
2. $P(n_0)$ is false. Then, we have to go forward and find the first $a \in \mathbb{N}$ for which $P(n_0 + a)$ is true.
   a) We find $a$ such that $P(n_0 + a)$ is true.
   b) We cannot find $a$ (after, say, a trillion iterations) where $P(n_0 + a)$ is true.
2. \( P(n_0) \) is false. Then, we have to go forward and find the first \( a \in \mathbb{N} \) for which \( P(n_0 + a) \) is true.
   a) We find \( a \) such that \( P(n_0 + a) \) is true.

b) We cannot find \( a \) (after, say, a trillion iterations) where \( P(n_0 + a) \) is true.
   - What could this mean?
Case #2 in Detail

2. \( P(n_0) \) is false. Then, we have to go forward and find the first \( a \in \mathbb{N} \) for which \( P(n_0 + a) \) is true.
   
a) We find \( a \) such that \( P(n_0 + a) \) is true.

   b) We cannot find \( a \) (after, say, a trillion iterations) where \( P(n_0 + a) \) is true.
      
      • What could this mean?

      Either we have to try harder....
2. $P(n_0)$ is false. Then, we have to go forward and find the first $a \in \mathbb{N}$ for which $P(n_0 + a)$ is true.

a) We find $a$ such that $P(n_0 + a)$ is true.

b) We cannot find $a$ (after, say, a trillion iterations) where $P(n_0 + a)$ is true.

• What could this mean?

Or the theorem is bogus!
And Here’s Another

• Let $a$ be a sequence defined as follows:

$$a_n = \begin{cases} 
0, & n = 0 \\
2, & n = 1 \\
a_{\lfloor n/2 \rfloor} + a_{\lfloor n/4 \rfloor} + 5n, & n \geq 2 
\end{cases}$$

• Then, find $C \in \mathbb{R}$ such that

$$(\forall n \in \mathbb{N})[a_n \leq C \cdot n]$$
And Here’s Another

• Let $a$ be a sequence defined as follows:

$$a_n = \begin{cases} 
0, & n = 0 \\
2, & n = 1 \\
ad\left\lfloor \frac{n}{2} \right\rfloor + ad\left\lfloor \frac{n}{4} \right\rfloor + 5n, & n \geq 2
\end{cases}$$

• Then, find $C \in \mathbb{R}$ such that

$$(\forall n \in \mathbb{N})[a_n \leq C \cdot n]$$
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• Then, find $C \in \mathbb{R}$ such that

$$(\forall n \in \mathbb{N}) [a_n \leq C \cdot n]$$

• We proceed via strong induction on $n$.  

Recursions like this have linear upper bounds.
And Here’s Another

• Let $a$ be a sequence defined as follows:

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0, & n = 0 \\
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• Then, find $C \in \mathbb{R}$ such that 

$$(\forall n \in \mathbb{N})[a_n \leq C \cdot n]$$

• We proceed via strong induction on $n$.

• In fact, to make some of the math easier, we will assume the hypothesis until $P(n - 1)$ and prove the step for $P(n)$ instead of $P(n + 1)$
Finding C

• IB:
  • For \( n = 0 \), \( T_0 \leq C \cdot 0 \) \( \Rightarrow \) \( 0 \leq 0 \). No constraints on \( C \) yet!
  • For \( n = 1 \), \( T_1 \leq C \cdot n \) \( \Rightarrow \) \( 2 \leq C \). Done. We have our first lower bound for \( C \).

• IH: Let \( n \geq 2 \). Then, assume \((\forall i \in \{0,1,2,...,n-1\})[P(i)]\), where \( P(i) \) means \( a_i \leq C \cdot i \)

• IS: We attempt to prove \((P(0) \land P(1) \land P(2) \land \cdots \land P(n-1)) \Rightarrow P(n)\):

\[
\bigwedge_{i=0}^{i=n-1} (a_i \leq C \cdot i) \Rightarrow a_n \leq C \cdot n
\]
Finding C

• IS: We attempt to prove \((P(1) \land P(2) \land \cdots \land P(n - 1)) \Rightarrow P(n)\):

\[
\bigwedge_{i=0}^{i=n-1} (a_i \leq C \cdot i) \Rightarrow a_n \leq C \cdot n
\]

• From the IH, and taking into consideration that \(0 \leq \lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{2} \rfloor \leq n\), we have (next slide):
Finding C

• From the IH, and taking into consideration that $0 \leq \lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{2} \rfloor \leq n$, we have:

\[
\begin{align*}
    a_{\lfloor n/4 \rfloor} & \leq C \cdot \lfloor n/4 \rfloor \leq C \cdot \frac{n}{4} \\
    a_{\lfloor n/2 \rfloor} & \leq C \cdot \lfloor n/2 \rfloor \leq C \cdot \frac{n}{2}
\end{align*}
\]

• $a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/4 \rfloor} + 5n \leq C \cdot \frac{n}{2} + C \cdot \frac{n}{4} + 5n = \frac{n \cdot (3C + 20)}{4}$
Finding C

• We have:

\[ a_n \leq \frac{n(3C + 20)}{4} \]

• We want:

\[ a_n \leq C \cdot n \]

• Hence, we want a C such that:

\[ \frac{n \cdot (3C + 20)}{4} \leq C \cdot n \]
Finding C

\[
\frac{n(3C + 20)}{4} \leq C \cdot n^{n \geq 1} \iff \frac{4}{(3C + 20)} \leq C \iff \frac{4}{3C + 20} \leq 4C \iff C \geq 20
\]

⇒ \( C_{\text{min}} = 20 \)
Constraints

• From the IB: $C \geq 2$
• From the IS: $C \geq 20$
• Since we want to minimize $C$, we set $C = 20$. 
STOP
RECORDING