START RECORDING

Constructive Induction

CMSC 250

Introductory Example

- We already know that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

- **But how? Who told us this?**
- This is **not** how math works; we usually do **not** know the answer ahead of time!

Making a Good Guess with Calculus

- Calculus tells us that (discrete) sums are approximations of (continuous) integrals.
- Then, we can observe that:

$$\sum_{i=1}^{n} i \approx \int_{1}^{n} x\,dx = \frac{1}{2}n^2 + c, \qquad c \in \mathbb{R}$$

- So we know that the sum ought to be *some* quadratic function of n.

Making a Good Guess with CS

- Another way to guess the quadratic form would be with plotting!
- Suppose $f(n) = \sum_{i=1}^{n} i$. Then:
  - $f(0) = \sum_{i=1}^{0} i = 0$
  - $f(1) = \sum_{i=1}^{1} i = 1$
  - $f(2) = \sum_{i=1}^{2} i = 1 + 2 = 3$
  - $f(3) = \sum_{i=1}^{3} i = 1 + 2 + 3 = 6$
  - …
  - $f(2) = \sum_{i=1}^{30} i = 1 + 2 + \cdots + 30 = 465$
- We can then fit a curve and see the quadratic curve by ourselves!

Making a Good Guess

- We saw that the sum is some quadratic polynomial. This is all we know!
- So $\sum_{i=1}^{n} i$ is some $poly(n)$ with degree 2, i.e

$$\sum_{i=1}^{n} i = An^2 + Bn + C, \qquad A, B, C \in \mathbb{R}$$

- ***How to determine A, B, and C?***

General Logic

- Solve **as if** you had an inductive proof (so IB, IH, IS)
- For every step, we will establish **conditions** on A, B,C **such that** the relevant step is correct.
  - Contrast this with **directly proving** that every step is correct.

Constant C

- IB: LHS is $\sum_{i=1}^{0} i = 0$. For RHS to be equal to LHS we need:

$$An^2 + Bn + C = 0 \Rightarrow C = 0$$

- So we already know that $C = 0$.

Co-efficients A, B

- IH: Assume that the proposition holds for $n \geq 0$. Then:

$$\sum_{i=1}^{n} i = An^2 + Bn$$

- IS: We want to prove that

$$\left(\sum_{i=1}^{n} i = An^2 + Bn\right) \Rightarrow \left(\sum_{i=1}^{n+1} i = A(n+1)^2 + B(n+1)\right)$$

Co-efficients A, B

- IH: Assume that the proposition holds for $n \geq 0$. Then:

$$\sum_{i=1}^{n} i = An^2 + Bn \quad \rightarrow P(n)$$

- IS: We want to prove that

$$\underbrace{\left(\sum_{i=1}^{n} i = An^2 + Bn\right)}_{P(n)} \Rightarrow \underbrace{\left(\sum_{i=1}^{n+1} i = A(n+1)^2 + B(n+1)\right)}_{P(n+1)}$$

Co-efficients A, B

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) \overset{\text{IH}}{=} An^2 + Bn + (n+1)$$

- We have to equate this to $A(n+1)^2 + B(n+1)$, since this is what we're trying to prove:

$$An^2 + Bn + (n+1) = A(n+1)^2 + B(n+1) \Rightarrow$$
$$\cancel{An^2} + \cancel{Bn} + (n+1) = \cancel{An^2} + 2An + A + \cancel{Bn} + B \Rightarrow$$
$$n + 1 = 2An + (A + B)$$

Co-efficients A, B

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- This is an equality between polynomials of k, so equating the co-efficients yields:

$$1 = 2A$$
$$A + B = 1$$

Co-efficients A, B

$$n + 1 = 2An + (A + B)$$

- This is an equality between polynomials in n, so equating the co-efficients yields:

$$1 = 2A$$
$$A + B = 1$$

- Note: The IS did not end up with **TRUE**, but with conditions on A,B **for it to be TRUE.**

All Our Constraints

1. $C = 0$
2. $A + B = 1$
3. $2 \cdot A = 1$

- Algebra yields $A = B = {}^{1}/_{2}$, so:

$$\sum_{i=0}^{n} i = \frac{1}{2} n^2 + \frac{1}{2} n + 0 = \frac{n(n+1)}{2}$$

What if Our Guess is Wrong (Over)?

1. Suppose we guess

$$\sum_{i=1}^{n} i = A \cdot n^3 + B \cdot n^2 + C \cdot n + D$$

2. **This still works**, we will just find $A = 0$ (try it at home!)

What if Our Guess is Wrong (Under)?

1. Suppose we guess

$$\sum_{i=1}^{n} i = A \cdot n + B$$

2. **This does not work (infeasible equation)**, no $A, B \in \mathbb{R}$ will satisfy the constraints (try it at home!)

Another Example (with Bounds!)

- Let a be a sequence defined as follows:

$$a_n = \begin{cases} 2, & n = 0 \\ 50, & n = 1 \\ 10a_{n-1} + 3a_{n-2}, & n \geq 2 \end{cases}$$

- Task: Find an upper bound for a_n.

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- What kind of inductive structure am I expecting?

Weak

Strong

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Weak

Strong

An inductive base with > 1 elements and a recursive rule with references to two prior terms hints towards strong induction…

Key Step

$$a_n = \begin{cases} 2, & n = 0 \\ 50, & n = 1 \\ 10a_{n-1} + 3a_{n-2}, & n \geq 2 \end{cases}$$

- Because of our experience with sequences like Fibonacci, Tribonacci that all have this form, we **suspect**:

$$a_n \leq C \cdot D^n, \quad C, D \in \mathbb{R}$$

Constraints on C

- IB:
  - $a_0 \leq C \cdot D^0 \Leftrightarrow 2 \leq C$
  - $a_1 \leq C \cdot D^1 \Leftrightarrow 50 \leq C \cdot D$

Inductive Hypothesis

- IB:
  - $a_0 \leq C \cdot D^0 \Leftrightarrow 2 \leq C$
  - $a_1 \leq C \cdot D^1 \Leftrightarrow 50 \leq C \cdot D$
- IH: Let $n \geq 1$. Assume that $(\forall i \in \{0, 1, 2, \dots n\})[a_i \leq C \cdot D^i]$

Inductive Step

- IB:
  - $a_0 \leq C \cdot D^0 \Leftrightarrow 2 \leq C$
  - $a_1 \leq C \cdot D^1 \Leftrightarrow 50 \leq C \cdot D$
- IH: Let $n \geq 1$. Assume that $\forall i \in \{0, 1, 2, \ldots n\},\ a_i \leq C \cdot D^i$.
- IS:

$$(\forall i \in \{0, 1, 2, \ldots n\})[a_i \leq C \cdot D^i] \Rightarrow (a_{n+1} \leq C \cdot D^{n+1})$$

Inductive Step

- IS:

$$(\forall i \in \{0,1,2,\dots n\})[a_i \leq C \cdot D^i] \Rightarrow (a_{n+1} \leq C \cdot D^{n+1})$$

- From the definition of a, we have $a_{n+1} = 10a_n + 3a_{n-1}$. Therefore,

$$a_{n+1} = 10a_n + 3a_{n-1} \leq 10 \cdot C \cdot D^n + 3 \cdot C \cdot D^{n-1} \text{ (By IH)}$$

- Want $10 \cdot C \cdot D^n + 3 \cdot C \cdot D^{n-1} \leq C \cdot D^{n+1}$

Inductive Step

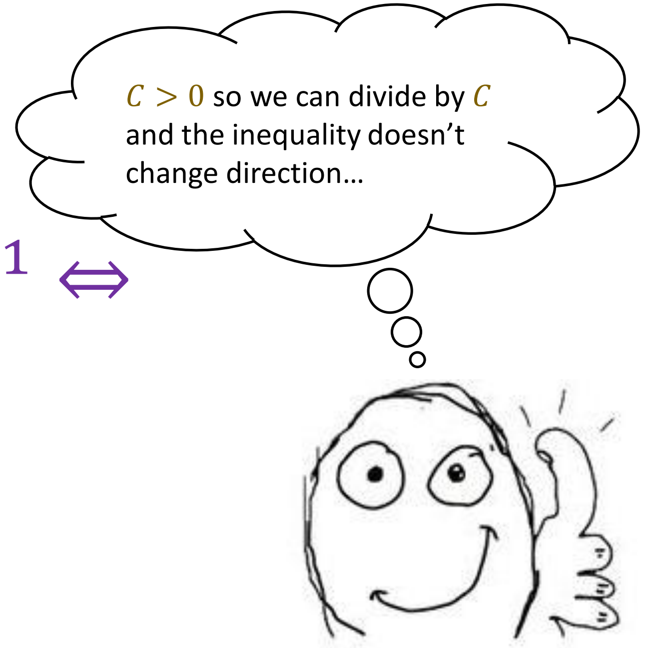
- Want

$$10 \cdot \cancel{C} \cdot D^n + 3 \cdot \cancel{C} \cdot D^{n-1} \leq \cancel{C} \cdot D^{n+1} \Leftrightarrow$$

$$10 \cdot D^n + 3 \cdot D^{n-1} \leq D^{n+1}$$

- Dividing both sides by D^{n-1} yields:

$$10D + 3 \leq D^2$$



All Constraints

1. $2 \leq C$
2. $50 \leq C \cdot D$
3. $10D + 3 \leq D^2$

- We deal with constraint 3 first.
  - Smallest $D \in \mathbb{R}^{>0}$ that satisfies it:

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  - Smallest $D \in \mathbb{N}$ that satisfies it: $D = \cdots ???$ (FIND ONE REAL QUICK, PLZ)

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$D = 11$ works! ☺

All Constraints

1. $2 \leq C$
2. $50 \leq C \cdot D$
3. $10D + 3 \leq D^2$

- Constraint (3) satisfied when $D \geq 11$ (just discussed)
- Since we want to find **tight** bounds for a_n, to minimize C, we select

$D = 11$ and from constraint (2) we have: $50 \leq C \cdot 11 \Leftrightarrow C \geq 4.55 \Rightarrow C_{min} = 4.55$

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- Conclusion:

$$a_n \leq 4.55 \cdot 11^n$$

Work on This

- A slight modification on the previous sequence:

$$a_n = \begin{cases} 10, & n = 0 \\ 50, & n = 1 \\ 10a_{n-1} + 3a_{n-2}, & n \geq 2 \end{cases}$$

- Assuming that we still suspect $a_n \leq C \cdot D^n$, **you** solve for the new C, D **right now**!

Work on This

- A slight modification on the previous sequence:

$$a_n = \begin{cases} 10, & n = 0 \\ 50, & n = 1 \\ 10a_{n-1} + 3a_{n-2}, & n \geq 2 \end{cases}$$

- Assuming that we still suspect $a_n \leq C \cdot D^n$, solve for the new C, D!
- Your solution ought to be $C = 10, D = 11$. What do you observe?

Coin Problem

- In Celestia, there are only $7c$ and $10c$ coins.
- We want to find the *least monetary amount* payable **exclusively** with such coins!
- In quantifiers (all quantifications assumed over $\mathbb{N}$)

$$(\forall n \geq A)(\exists n_1, n_2)[n = 7n_1 + 10n_2]$$

- **Goal: Find constraints on *A* via constructive induction!**
- IB: ???

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- **Goal: Find constraints on *A* via constructive induction!**
- IB: Defer for later!!!

- IH: Assume that for $n \geq A$, $(\exists n_1', n_2')[n = 7 \cdot n_1' + 10n_2']$

Coin Problem (IS)

- From the IH we have $(\exists n_1', n_2')[n = 7 \cdot n_1' + 10n_2']$
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- How can we add/remove coins to get another cent?
  1. $n_2' \geq 2$: Remove two $10c$ coins, add three $7c$ coins

$$n + 1 = 7n_1' + 10n_2' + 1 = 7n_1' + 10n_2' + (21 - 20)$$
$$= 7(n_1' + 3) + 10(n_2' - 2)$$

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$$= 7(n_1' + 3) + 10(n_2' - 2)$$

2. $n_1' \geq 7$: Remove seven $7c$ coins, add five $10c$ coins

$$n + 1 = 7n_1' + 10n_2' + 1 = 7n_1' + 10n_2' + (50 - 49)$$
$$= 7(n_1' - 7) + 10(n_2' + 5)$$

Coin Problem (IS)

3. $(n_1' \leq 6) \wedge (n_2' \leq 1)$: Max value is $6 \times 7 + 1 \times 10 = 52$, so $n \leq 52$.

RECAP

- We've shown that if $n \geq 53$, then

$$((\exists n_1', n_2')[n = 7 \cdot n_1' + 10n_2']) \Rightarrow ((\exists n_1'', n_2'')[n + 1 = 7 \cdot n_1'' + 10n_2''])$$

- For which n do we know that $((\exists a, b \in \mathbb{N})[n = 7a + 10b]$?

$\forall n \geq 52$

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Something Else

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- We've shown that if $n \geq 53$, then

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Something Else

*Only the implication holds! We don't have any **hard truth** (base) about whether it EVER holds.*

Coin Problem (IS)

3. $(n_1' \leq 6) \wedge (n_2' \leq 1)$: Max value is $6 \times 7 + 1 \times 10 = 52$, so $n \leq 52$.

- Condition: $A \geq 53$.
- **Now** I need a base case.
- $(\exists ?\, n_1'', n_2'' \in \mathbb{N})[53 = 7 \cdot n_1'' + 10 n_2'']$

Yes (which?)

No

Coin Problem (IS)

3. $(n_1' \leq 6) \wedge (n_2' \leq 1)$: Max value is $6 \times 7 + 1 \times 10 = 52$, so $k \leq 52$.

- Condition: $A \geq 53$.
- **Now** I need a base case.
- $(\exists ? n_1'', n_2'' \in \mathbb{N})[53 = 7 \cdot n_1'' + 10 n_2'']$

Yes (which?)

No

Prove it at home (use cases)

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Yes (which?)

No

$n_1'' = 2,$
$n_2'' = 4$

RECAP

- We've shown that if $n \geq 53$, then

$$((\exists n_1', n_2')[n = 7 \cdot n_1' + 10n_2']) \Rightarrow ((\exists n_1'', n_2'')[n + 1 = 7 \cdot n_1'' + 10n_2''])$$

- We've also shown that $(\exists r_1, r_2 \in \mathbb{N})[54 = 7r_1 + 10r_2]$

$$(r_1 = 2, r_2 = 4)$$

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- We've shown that if $n \geq 53$, then

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$(r_1 = 2, r_2 = 4)$
- What do we know NOW about the theorem?

True for $n \geq 52$

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- We've shown that if $n \geq 53$, then

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- What do we know NOW about the theorem?

True for $n \geq 52$	True for $n \geq 53$	True for $n \geq 54$	Nothing

What is A?

- Recall the theorem (all quantifiers over $\mathbb{N}$):

$$(\forall n \geq A)(\exists n_1, n_2)[n = 7n_1 + 10n_2]$$

- Our goal was to find A.
- $A = 54$ works, and is **optimal, since** $A = 53$ does not work.

Question

- Is the theorem true for any $n \leq 53$?

Yes
(which?)

No
(Why?)

Question

- Is the theorem true for any $n \leq 53$?

Yes (which?)

No (Why?)

$$0, 7, 10, 14, 17, 20, 21, 24, 27, 28, 30, 31, 34, 35, 37, 38, 40, 41, 42, 44, 45, 47, 48, 49, 50, 51, 52$$

- Note that there are **gaps** between these integers!

General Scenarios

- Once we establish

$$(\forall n \geq n_0)[P(n) \Rightarrow P(n+1)]$$

  we have two cases:

1. $P(n_0)$ is true. Then, we have to go back and find the first $a \in \mathbb{N}$ for which $P(n_0 - a)$ is **false.** This means that $A = n_0 - a + 1$

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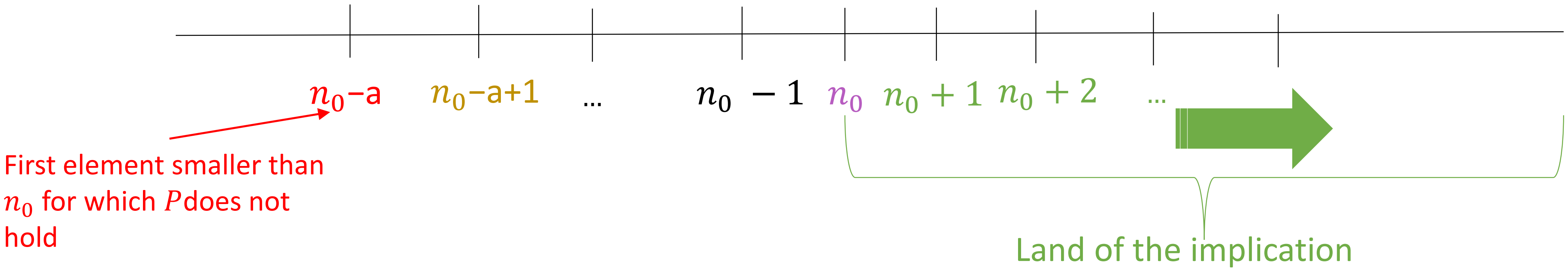

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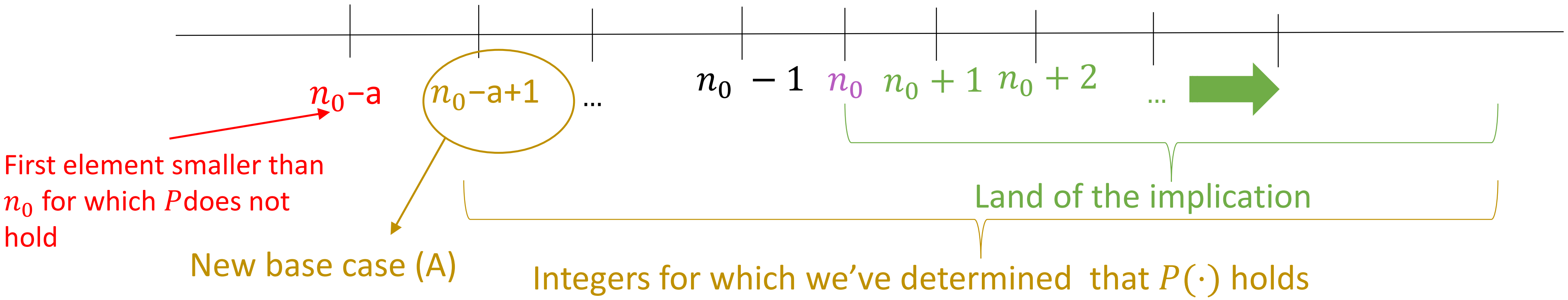

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- Once we establish

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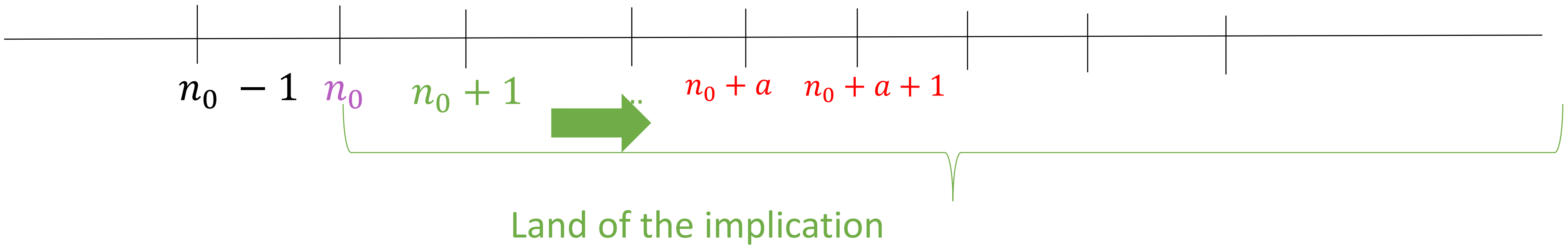
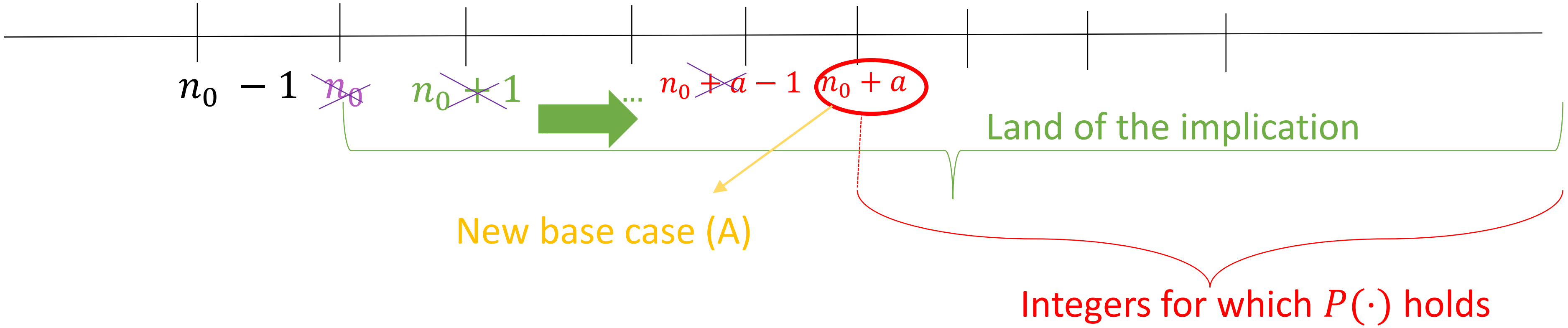
we have a second case:

2. $P(n_0)$ is **false**. Then, we have to go **forward** and find the first $a \in \mathbb{N}$ for which $P(n_0 + a)$ is **true**. This means that $A = n_0 + a$



Case #2

2. $P(n_0)$ is **false.** Then, we have to go **forward** and find the first $a \in \mathbb{N}$ for which $P(n_0 + a)$ is **true.**
   a) We find a such that $P(n_0 + a)$ is **true.**



Case #2 in Detail

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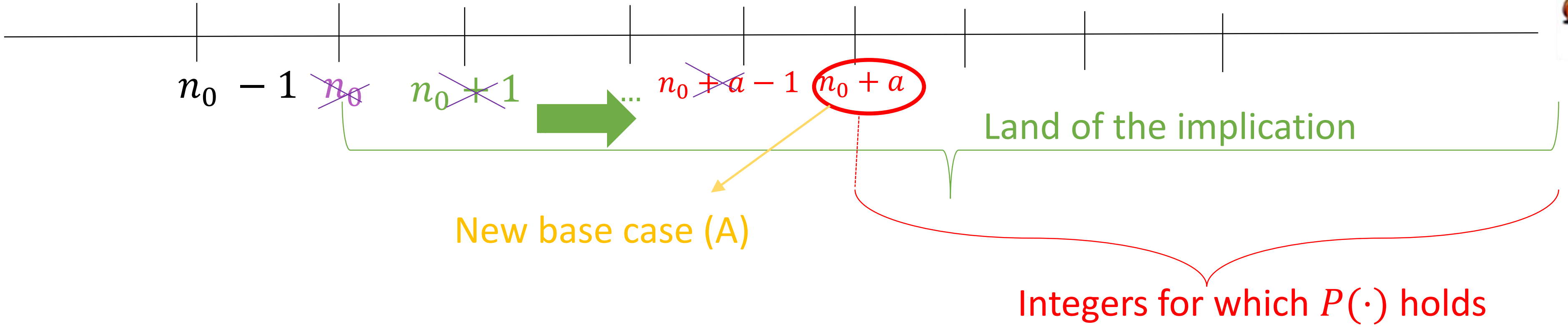

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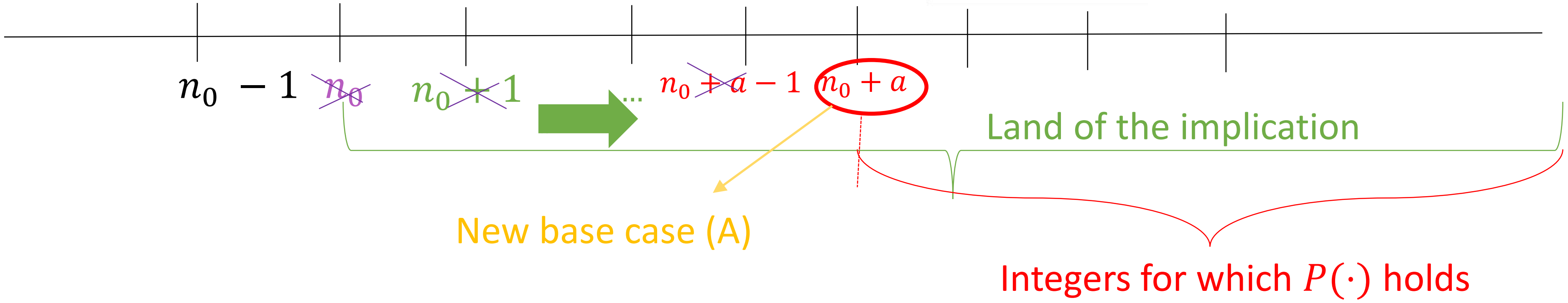

   b) We cannot find a (after, say, a trillion iterations) where $P(n_0 + a)$ is **true**.

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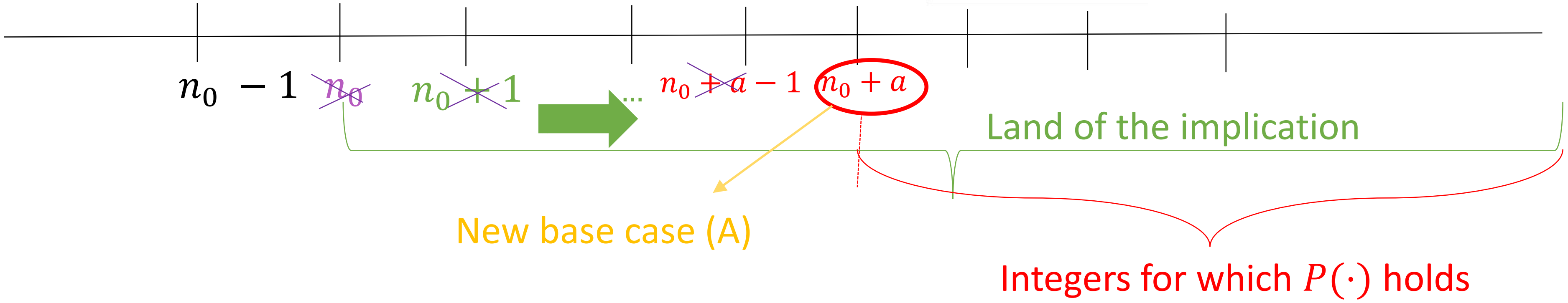

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- **What could this mean?**

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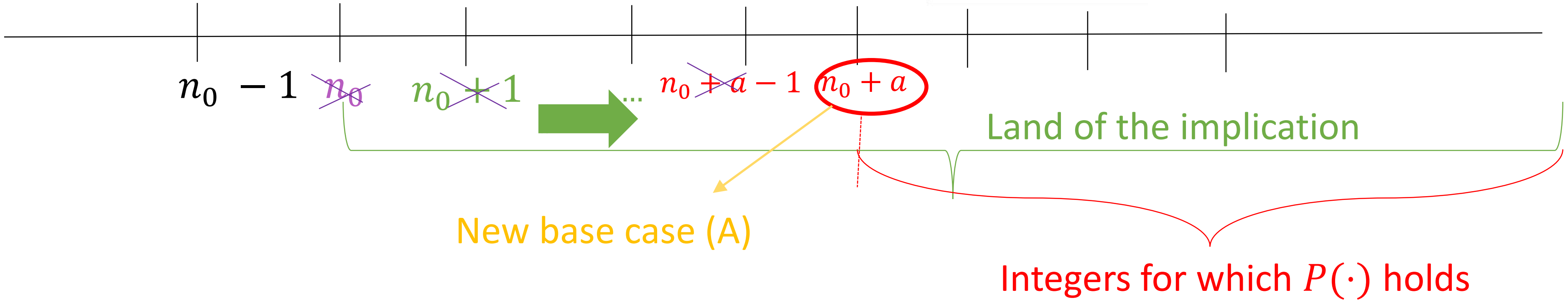

b) We cannot find a (after, say, a trillion iterations) where $P(n_0 + a)$ is **true**.

- **What could this mean?**

Either we have to try harder….

Case #2 in Detail

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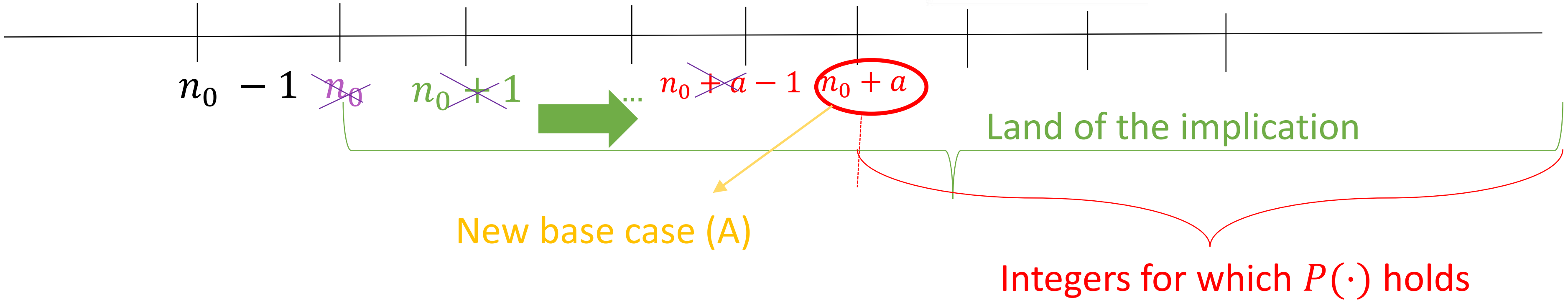

   b) We cannot find a (after, say, a trillion iterations) where $P(n_0 + a)$ is **true**.
      - **What could this mean?**

Or the theorem is bogus!

And Here's Another

- Let a be a sequence defined as follows:

$$a_n = \begin{cases} 0, & n = 0 \\ 2, & n = 1 \\ a_{\left\lfloor \frac{n}{2} \right\rfloor} + a_{\left\lfloor \frac{n}{4} \right\rfloor} + 5n, & n \geq 2 \end{cases}$$

- Then, find $C \in \mathbb{R}$ such that

$$(\forall n \in \mathbb{N})[a_n \leq C \cdot n]$$

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- We proceed via **strong induction** on n.

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- We proceed via **strong induction** on n.
- In fact, to make some of the math easier, we will assume the hypothesis until $P(n-1)$ and prove the step for $P(n)$ instead of $P(n+1)$

Finding C

- IB:
  - For $n = 0, T_0 \leq C \cdot 0 \Leftrightarrow 0 \leq 0$. No constraints on C yet!
  - For $n = 1, T_1 \leq C \cdot n \Leftrightarrow 2 \leq C$. Done. We have our first lower bound for C.
- IH: Let $n \geq 2$. !!!! Then, assume $(\forall i \in \{0,1,2,\ldots,n-1\})[P(i)]$, where $P(i)$ means $a_i \leq C \cdot i$
- IS: We attempt to prove $(P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(n-1)) \Rightarrow P(n)$:

$$\bigwedge_{i=0}^{i=n-1} (a_i \leq C \cdot i) \Rightarrow a_n \leq C \cdot n$$

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- From the IH, and taking into consideration that $0 \leq \left\lfloor \frac{n}{4} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor \leq n$, we have (next slide):

Finding C

- From the IH, and taking into consideration that $0 \leq \left\lfloor \frac{n}{4} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor \leq n$, we have:

$$\begin{cases} a_{\lfloor n/4 \rfloor} \leq C \cdot \lfloor n/4 \rfloor \leq C \cdot \frac{n}{4} \\ a_{\lfloor n/2 \rfloor} \leq C \cdot \lfloor n/2 \rfloor \leq C \cdot \frac{n}{2} \end{cases}$$

- $a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/4 \rfloor} + 5n \leq C \cdot \frac{n}{2} + C \cdot \frac{n}{4} + 5n = \frac{n*(3C+20)}{4}$

Finding C

- We have:

$$a_n \leq \frac{n*(3C+20)}{4}$$

- We want:

$$a_n \leq C \cdot n$$

- Hence, we want a C such that:

$$\frac{n*(3C+20)}{4} \leq C \cdot n$$

Finding C

$$\frac{\cancel{n}(3C + 20)}{4} \leq C \cdot \cancel{n} \overset{n \geq 1}{\Leftrightarrow}$$

$$\frac{(3C + 20)}{4} \leq C \Leftrightarrow$$

$$3C + 20 \leq 4C \Leftrightarrow$$

$$C \geq 20$$

$$\Rightarrow C_{min} = 20$$

Constraints

- From the IB: $C \geq 2$
- From the IS: $C \geq 20$
- Since we want to minimize C, we set $C = 20$.

STOP RECORDING