Duplicator-Spoiler Games

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We will call SPOIL S and DUP D to fit on slides.

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Bill plays a student $(L_3, L_4, 2)$, $(L_3, L_4, 3)$

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1. S beats D in the (L_a, L_b, k) game.
2. D beats S in the $(L_a, L_b, k-1)$ game.

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1. Who wins $(L_3, L_4, 2)$? (2 moves).
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3. GENERALLY: Who wins (L_a, L_b, k).

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Play a student $\mathbb{N}$ and $\mathbb{Z}$ with 1 move, 2 moves

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3. D wins $(\mathbb{Z}, \mathbb{Q}, k-1)$, S wins $(\mathbb{Z}, \mathbb{Q}, k)$.
4. D wins $(L_{10}, \mathbb{N} + \mathbb{N}^*, k-1)$, S wins $(L_{10}, \mathbb{N} + \mathbb{N}^*, k)$.

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4. D wins $(L_{10}, \mathbb{N}+\mathbb{N}^*, k-1)$, S wins $(L_{10}, \mathbb{N}+\mathbb{N}^*, k)$.
5. D wins $(\mathbb{N}+\mathbb{Z}, \mathbb{N}, k-1)$, S wins $(\mathbb{N}+\mathbb{Z}, \mathbb{N}, k)$.

A Notion of L, L' being Similar

Let L and L' be two linear orderings.

Definition

If D wins the k-round DS-game on L, L' then L, L' are *k-game equivalent* (denoted $L \equiv_k^G L'$).

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If $Q \in \{\exists, \forall\}$ then

$$\mathrm{qd}((Qx_1)[\phi(x_1, \ldots, x_n)] = \mathrm{qd}(\phi_1(x_1, \ldots, x_n)) + 1.$$

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$$\mathrm{qd}((\forall x)(\forall z)[x < z \rightarrow (\exists y)[x < y < x]]) = 2 + 1 = 3$$

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L and L' are *k-truth-equiv* ($L \equiv_k^T L'$)

$$(\forall \phi, qd(\phi) \le k)[L \models \phi \text{ iff } L' \models \phi.$$

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2. Well foundedness cannot be expressed in first order at all! (Proof: $(\forall n)[\mathbb{N} + \mathbb{Z} \equiv_n^G \mathbb{N})$.
3. Upshot: Questions about expressability become questions about games.